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Existence et stabilité de solutions fortes en théorie cinétique des gaz

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Existence et stabilité de solutions fortes en théorie cinétique des gaz

Isabelle Tristani

Existence et stabilité de solutions fortes en théorie cinétique des gaz

Résumé. Cette thèse est centrée sur l'étude d'équations issues de la théorie cinétique des gaz. Dans tous les problèmes qui y sont explorés, une analyse des problèmes linéaires ou linéarisés associés est réalisée d'un point de vue spectral et du point de vue des semi-groupes. A cela s'ajoute une analyse de la stabilité non linéaire lorsque le modèle est non linéaire. Plus précisément, dans une première partie, nous nous intéressons aux équations de Fokker-Planck fractionnaire et Boltzmann sans cut-off homogène en espace et nous prouvons un retour vers l'équilibre des solutions de ces équations avec un taux exponentiel dans des espaces de type L^1 à poids polynomial. Concernant l'équation de Landau inhomogène en espace, nous développons une théorie de Cauchy de solutions perturbatives dans des espaces de type L^2 avec différents poids (polynomiaux ou exponentiels) et nous prouvons également la stabilité exponentielle de ces solutions. Nous démontrons ensuite pour l'équation de Boltzmann inélastique inhomogène avec terme diffusif le même type de résultat dans des espaces L^1 à poids polynomial dans un régime de faible inélasticité. Pour finir, nous étudions dans un cadre général et uniforme des modèles qui convergent vers l'équation de Fokker-Planck du point de vue de l'analyse spectrale et des semi-groupes.

Mots-clés : théorie cinétique ; équation de Boltzmann ; collisions inélastiques ; équation de Boltzmann sans cut-off ; équation de Landau ; potentiels durs ; potentiels faiblement mous ; équation de Fokker-Planck ; diffusion fractionnaire ; retour à l'équilibre ; convergence exponentielle ; trou spectral ; décroissance du semi-groupe ; hypodissipativité.

Existence and stability of strong solutions in kinetic theory

Abstract. The topic of this thesis is the study of models coming from kinetic theory. In all the problems that are addressed, the associated linear or linearized problem is analyzed from a spectral point of view and from the point of view of semigroups. To that, we add the study of the nonlinear stability when the equation is nonlinear. More precisely, to begin with, we treat the problem of trend to equilibrium for the fractional Fokker-Planck and Boltzmann without cut-off equations, proving an exponential decay to equilibrium in spaces of type L^1 with polynomial weights. Concerning the inhomogeneous Landau equation, we develop a Cauchy theory of perturbative solutions in spaces of type L^2 with various weights such as polynomial and exponential weights and we also prove the exponential stability of these solutions. Then, we prove similar results for the inhomogeneous inelastic diffusively driven Boltzmann equation in a small inelasticity regime in L^1 spaces with polynomial weights. Finally, we study in the same and uniform framework from the spectral analysis point of view with a semigroup approach several Fokker-Planck equations which converge towards the classical one.

Keywords. kinetic theory ; Boltzmann equation ; inelastic collisions ; Boltzmann equation without cut-off ; Landau equation ; hard potentials ; moderately soft potentials ; Fokker-Planck equation ; fractional diffusion ; trend to equilibrium ; exponential convergence ; spectral gap ; semigroup decay ; hypodissipativity.

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Table des matières

Introduction générale

1 La théorie cinétique des gaz

L'objet de la théorie cinétique des gaz est de modéliser tout système formé d'un grand nombre de particules tel qu'un gaz. Pour un tel système, plusieurs niveaux de description sont envisageables. Une manière de décrire ce système consiste en l'étude des trajectoires de chaque particule, c'est la *description microscopique*. Une telle description présente néanmoins des inconvénients, d'une part du fait du grand nombre de particules considérées, et d'autre part car une telle description ne donne pas accès aux grandeurs physiques intéressantes dites observables que sont la masse, la vitesse moyenne et la température. Une autre façon de procéder serait de décrire les grandeurs macroscopiques de ce système (quantités évoquées précédemment que l'on peut effectivement mesurer), c'est la *description macroscopique*. La *théorie cinétique* se situe précisément à un niveau intermédiaire entre ces deux descriptions, microscopique et macroscopique que l'on appelle la *description mésoscopique*. Il s'agit d'une description statistique dont le but est de décrire le comportement “typique” d'une particule, permettant ainsi de simplifier l'étude détaillée des trajectoires de chaque particule tout en préservant les informations physiques du système. Cette théorie fournit également un cadre à l'étude des mécanismes de retour vers l'équilibre, une des finalités étant de prédire le comportement en temps long des systèmes étudiés. Ces différentes descriptions sont résumées dans le schéma suivant :

Description microscopique
Description des trajectoires de chaque particule
Equations de Newton

Description mésoscopique
Description de l'évolution de la densité de particules
Equations de Boltzmann, Landau, Fokker-Planck, Vlasov...

Description macroscopique
Description de l'évolution des observables (masse, vitesse, température)
Equations d'Euler, de Navier-Stokes

Dans cette thèse, nous nous plaçons dans le cadre d'une description mésoscopique, c'est-à-dire que l'on considère que le système de particules décrit se rapproche d'un continuum puisqu'il est constitué d'un très grand nombre de particules. L'état du système est alors décrit par une densité de particules $f = f(t,x,v)$ où $t \in \mathbb{R}^+$ est le temps, $x \in \Omega$ (où Ω est un domaine de $\mathbb{R}^3$) la position et $v \in \mathbb{R}^3$ la vitesse, $f(t,x,v)\,dx\,dv$ représente alors la quantité de particules dans l'élément de volume $dx\,dv$ centré en $(x,v) \in \Omega \times \mathbb{R}^3$.

Notons que dans certains cas, afin de simplifier l'étude, on ne prend pas en compte la dépendance spatiale de la densité, c'est-à-dire que l'on considère que la densité est homogène en espace et ne dépend donc que du temps et de la vitesse $f = f(t,v)$ (en général, l'étude du cas dit homogène en espace est l'étape préalable à l'étude complète du problème).

Pour les problèmes inhomogènes en espace, plusieurs cadres d'études existent selon le domaine Ω considéré. On peut considérer le cas de l'espace tout entier $\Omega = \mathbb{R}^3$, le cas du tore $\Omega = \mathbb{T}^3$ qui est un domaine borné sans bord ou le cas d'un ouvert convexe régulier Ω qui satisfait certaines propriétés au bord.

Comme annoncé précédemment, grâce à cette description, nous avons accès aux quantités macroscopiques observables que sont la masse locale ρ, la vitesse macroscopique locale u et la température T à l'instant $t \in \mathbb{R}^+$ et à la position x qui sont données par les formules suivantes :

$$\begin{gathered} \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad \rho\,u(t,x) = \int_{\mathbb{R}^3} v\,f(t,x,v)\,dv, \\ \rho\,T(t,x) = \frac{1}{3}\int_{\mathbb{R}^3} |v-u(t,x)|^2\,f(t,x,v)\,dv. \end{gathered} \tag{1}$$

1.1 Evolution de la densité de particules

Le but est donc de décrire l'évolution de la densité de particules f. D'après les lois de Newton, en l'absence de force extérieure ou d'interaction entre les particules, ces dernières se déplacent en ligne droite à vitesse constante :

$$v = \frac{dx}{dt}, \quad \frac{dv}{dt} = 0.$$

La densité f est alors solution de l'équation dite de transport libre :

$$\partial_t f + v \cdot \nabla_x f = 0.$$

Si maintenant, d'autres facteurs interviennent, cette équation doit être modifiée sous la forme

$$\partial_t f + v \cdot \nabla_x f = Q(f) \tag{2}$$

où nous expliciterons $Q(f)$ dans la suite selon le modèle considéré.

1.2 L'équation de Boltzmann

Parmi les équations de la théorie cinétique des gaz, l'*équation de Boltzmann* joue un rôle central, elle est utilisée pour modéliser un gaz raréfié hors équilibre. C'est en effet la plus ancienne des équations cinétiques dérivée formellement par Boltzmann [19] après des travaux précurseurs de Maxwell [62]. De plus, il s'agit de la seule équation pour laquelle il existe des théorèmes rigoureux qui permettent de l'établir à partir de la description microscopique. Pour dériver l'équation de Boltzmann, à partir d'un système de particules évoluant selon les lois de Newton, la bonne échelle est connue, c'est la limite de Boltzmann-Grad [47]. Une dérivation rigoureuse a été établie par Lanford [61] pour des temps courts. Revenons maintenant sur les hypothèses qui ont permis à Boltzmann de dériver son équation.

1. Les particules interagissent selon des *collisions binaires*, processus par lequel deux particules qui sont très proches l'une de l'autre voient leurs trajectoires respectives modifiées fortement en un temps très court. On suppose implicitement que le gaz considéré est assez dilué pour que l'on puisse négliger les collisions faisant intervenir plus de deux particules.

2. Les collisions sont *localisées en temps et en espace* : elles se produisent dans des échelles de temps et d'espace très petites par rapport aux échelles typiques de description.

3. Les collisions sont *élastiques* : il y a conservation de la quantité de mouvement et de l'énergie au cours d'une collision. Si l'on note v et v_* (respectivement v' et v'_* les vitesses de deux particules avant (respectivement après) collision, on a les lois suivantes :
$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2. \tag{3}$$
Nous obtenons donc (en dimension 3) quatre équations, les vitesses après collision v' et v'_* peuvent donc être paramétrées de la manière suivante :
$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v' = \frac{v - v_*}{2} + \frac{|v - v_*|}{2}\sigma \tag{4}$$
où σ est un élément de la sphère unité $\mathbb{S}^2$ (d'autres manières de paramétrer sont envisageables, c'est la "représentation σ" présentée ici que nous utiliserons).

4. Les collisions sont *microréversibles* : d'un point de vue probabiliste, la probabilité que les vitesses (v', v'_*) soient changées en (v, v_*) au cours d'une collision est la même que celle que les vitesses (v, v_*) soient changées en (v', v'_*).

5. L'hypothèse de *chaos moléculaire* de Boltzmann est satisfaite : les vitesses de deux particules qui sont sur le point d'entrer en collision ne sont pas reliées.

Sous ces hypothèses, Boltzmann [19] montre que l'équation générale (2) devient
$$\partial_t f + v \cdot \nabla_x f = Q_B(f, f)$$

où $Q_B(f,f)$ est un opérateur quadratique qui modélise les interactions entre particules et est donné par

$$Q_B(f,f)(t,x,v) := \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\left(f'f'_* - ff_*\right)\, dv_*\, d\sigma. \tag{5}$$

Nous utiliserons les notations standards suivantes : $f = f(t,x,v)$, $f' = f(t,x,v')$, $f_* = f(t,x,v_*)$, $f'_* = f(t,x,v'_*)$. La fonction B est appelée le noyau de collision de Boltzmann. Elle est positive, ne dépend que de la vitesse relative et du cosinus de l'angle de déviation. Selon le type de collisions considéré, B peut être explicité ou ne pas l'être, nous y reviendrons dans la présentation des modèles que nous étudions. Expliquons succinctement la forme de l'opérateur (5). Formellement, on peut le découper en deux parties, un terme de gain et un terme de perte :

$$Q_B(f,f) = Q_B^+(f,f) - Q_B^-(f,f).$$

Le terme de perte compte les collisions pendant lesquelles une particule de vitesse v rencontre une particule de vitesse v_*. Ce terme permet en quelque sorte de compter les collisions pour lesquelles une particule va en général perdre la vitesse v. D'autre part, lorsque deux particules de vitesses v' et v'_* entrent en collision, la particule de vitesse v' va en général acquérir la vitesse v, ce qui va augmenter le nombre de particules ayant la vitesse v, ceci est la signification du terme de gain.

Lois de conservation

Remarquons ici que les propriétés microscopiques conservatives (hypothèse 3.) des collisions se traduisent sur les grandeurs macroscopiques que sont la masse, la quantité de mouvement et l'énergie. On peut prouver formellement une formulation faible de l'opérateur de collision (en utilisant des changements de variables), si φ est une fonction test,

$$\int_{\mathbb{R}^3} Q_B(f,f)\,\varphi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\, ff_*\left(\varphi' + \varphi'_* - \varphi - \varphi_*\right)\, dv\, dv_*\, d\sigma,$$

de cette formulation, on déduit

$$\int_{\mathbb{R}^3} Q_B(f,f)\,\varphi(v)\,dv = 0, \quad \text{pour} \quad \varphi(v) = 1, v_1, v_2, v_3, |v|^2. \tag{6}$$

Dans le cas où $\Omega = \mathbb{T}^3$ (cadre dans lequel nous nous placerons pour les équations non homogènes en espace), en intégrant en $x \in \mathbb{T}^3$ les lois de conservation locales (6), nous obtenons la conservation de la masse globale, la quantité de mouvement globale et l'énergie globale :

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{T}^3} f(t,x,v)\,\varphi(v)\,dx\,dv = 0 \quad \text{pour} \quad \varphi(v) = 1, v_1, v_2, v_3, |v|^2.$$

Les lois de conservation locales (6) sont également à la base du passage entre la description mésoscopique et la description macroscopique. En effet, si $f = f(t,x,v)$ est solution de l'équation de Boltzmann, avec les notations (1), elles impliquent les relations suivantes :

$$\begin{aligned}
&\partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\
&\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x(\rho T) = -\mathrm{div}_x \int_{\mathbb{R}^3} A_1(v-u)\, f\, dv, \\
&\partial_t \left(\rho \left(\frac{1}{2}|u|^2 + \frac{3}{2}\, T \right) \right) + \mathrm{div}_x \left(\rho u \left(\frac{1}{2}|u|^2 + \frac{5}{2} T \right) \right) \\
&\quad = -\mathrm{div}_x \int_{\mathbb{R}^3} A_2(v-u)\, f\, dv - \mathrm{div}_x \int_{\mathbb{R}^3} A_1(v-u) \cdot u\, f\, dv,
\end{aligned} \tag{7}$$

où

$$A_1(z) := z \otimes z - \frac{1}{3}|z|^2, \quad A_2(z) = \frac{1}{2}(|z|^2 - 5) z.$$

Les membres de gauche des égalités précédentes (7) correspondent avec ceux d'une équation d'Euler compressible. En revanche, la partie droite des égalités (7) dépend de la solution f de l'équation de Boltzmann et n'est en général pas déterminée par les variables macroscopiques ρ, u et T. Néanmoins, dans certaines limites, il est possible d'approximer les membres de droite par des fonctions de ρ, u et T. Par exemple, dériver l'équation d'Euler comme une limite asymptotique de l'équation de Boltzmann revient à prouver que les membres de droite de (7) s'annulent dans cette limite. Il est également possible de dériver l'équation de Navier-Stokes dans une autre limite asymptotique de l'équation de Boltzmann.

Dissipation d'entropie

Toujours en utilisant des changements de variable, on peut donner une autre formulation faible de l'opérateur de collision (5) :

$$\int_{\mathbb{R}^3} Q_B(f,f)\, \varphi(v)\, dv = \\
\frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma)\, (f' f'_* - f f_*)\, (\varphi + \varphi_* - \varphi' - \varphi'_*)\; dv\, dv_*\, d\sigma.$$

Cette formulation amène à l'inégalité suivante :

$$D(f) := -\int_{\mathbb{R}^3} Q_B(f,f)\, \log(f)\, dv \geq 0. \tag{8}$$

Venons en maintenant aux conséquences de l'inégalité (8). Nous introduisons la fonctionnelle H de Boltzmann ou entropie :

$$H(f) := \int_{\mathbb{R}^3 \times \mathbb{T}^3} f\, \log f\, dx\, dv.$$

On déduit alors le Théorème H de Boltzmann :

$$\frac{d}{dt}H(f) = -\int_{\mathbb{T}^3} D(f)\,dx \leq 0. \tag{9}$$

De plus, une distribution f vérifiant $D(f)(t,x) = 0$ est une maxwellienne et plus précisément, est de la forme :

$$\forall\, v \in \mathbb{R}^3, \quad f(t,x,v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{3/2}} e^{-\frac{|u(t,x)-v|^2}{2T(t,x)}} =: M(\rho,u,T)(v),$$

où nous utilisons toujours les notations (1). On dit alors que f est un équilibre thermodynamique local. Pour déterminer les équilibres globaux, il faut alors chercher les maxwelliennes M vérifiant

$$\forall\, x \in \mathbb{T}^3, \quad \forall\, v \in \mathbb{R}^3, \quad v \cdot \nabla_x M = 0.$$

On trouve des maxwelliennes M dépendant uniquement de la variable v. De plus, si la masse totale, la quantité de mouvement totale et l'énergie totale sont fixées, M est définie de manière unique.

1.3 Autres équations cinétiques

Nous présentons ici d'autres équations qui sont en quelque sorte des variantes de l'équation de Boltzmann précédemment introduite, qui est, comme on l'a dit, pour des raisons historiques et mathématiques, un modèle central en théorie cinétique des gaz.

Diverses équations de Boltzmann

Tout d'abord, rappelons que selon le type de collisions considérées, la forme du noyau de collision B est différente, ce qui conduit de fait à différents modèles. Les collisions usuelles considérées sont les collisions de type sphères dures ou des collisions selon un potentiel d'interaction. Dans ce dernier cas, soulignons le caractère non intégrable sur la sphère du noyau de collision B. Cette non intégralité provient d'une singularité qui apparait pour les petits angles de déviation c'est-à-dire pour les collisions dites rasantes. Ceci induit des difficultés liées à l'impossibilité de découper l'opérateur de collision en deux parties, terme de gain et terme de perte. Grad a alors introduit son hypothèse dite de *troncature angulaire de Grad* (voir [48]) ou de cut-off qui consiste à "couper" les collisions rasantes afin de supprimer la singularité en 0 de la partie angulaire du noyau. L'étude de l'équation sans hypothèse simplificatrice a été abordée plus tardivement.

Une variante des équations de Boltzmann que l'on vient de mentionner est une équation de Boltzmann dite *inélastique*. Cette équation est utilisée pour modéliser des gaz granulaires. L'hypothèse d'élasticité des équations est alors supprimée et on considère qu'une perte d'énergie s'opère lors des collisions. Les lois physiques de la collision se résument alors à la conservation de la quantité de mouvement : $v + v_* = v' + v'_*$. Une hypothèse classique sur les interactions binaires entre les grains est celle de collisions de sphères dures inélastiques, sans perte de vitesse relative tangentielle (selon la direction de l'impact) et une perte de vitesse relative normale. Cette perte est représentée et quantifiée par un coefficient dit de restitution qui intervient dans la forme l'opérateur de collision noté ici $Q_{B_{in}}$. Si l'on note v et v_* (resp. v' et v'_*) les vitesses de deux particules avant (resp. après) la collision, les vitesses post collision sont données par

$$v' = v - \frac{1+e_\lambda}{2}\left(\left(v - v_*\right)\cdot \widehat{n}\right)\widehat{n}, \quad v'_* = v_* + \frac{1+e_\lambda}{2}\left(\left(v - v_*\right)\cdot \widehat{n}\right)\widehat{n},$$

où e_λ est le coefficient de restitution, et $\widehat{n}$ détermine l'impact de la direction. Ceci conduit à une équation de la forme

$$\partial_t f + v \cdot \nabla_x f = Q_{B_{in}}(f, f).$$

Si l'on considère cette équation, on voit apparaitre un phénomène de “gel”, les particules ont tendance à acquérir la vitesse macroscopique locale, phénomène provoqué par la dissipation d'énergie se produisant au moment des collisions. Au niveau macroscopique, on voit apparaitre un phénomène de “clustering” et plusieurs équilibres sont alors créés lorsque l'on considère le problème non homogène en espace. Un autre modèle à étudier est

$$\partial_t f + v \cdot \nabla_x f = Q_{B_{in}}(f, f) + \Delta_v f,$$

où l'on a ajouté un terme diffusif modélisant une excitation de type secousse ou chaleur par exemple, le but étant d'empêcher au maximum le phénomène de “clustering”. En effet, l'ajout de ce terme induit une augmentation de la température. Cette équation conserve la masse et la quantité de mouvement mais pas l'énergie.

Equation de Landau

Parlons maintenant d'une autre équation qui est dérivée de l'équation de Boltzmann, l'*équation de Landau*. C'est un modèle utilisé en physique des plasmas. Landau a établi dans [59] par des arguments formels une asymptotique du noyau de Boltzmann dans le régime de collisions rasantes dominantes. Il s'agit de rendre toutes les collisions rasantes, plus précisément, en introduisant des opérateurs de collision Q_{B_ε} dont la partie angulaire se concentre sur les collisions rasantes lorsque $\varepsilon \to 0$, on dérive l'équation de Landau de l'équation de Boltzmann. On cite ici l'article d'Alexandre et Villani [2] dans lequel on peut voir que l'approximation de Landau est désormais comprise mathématiquement dans un cadre dit d'asymptotique des collisions rasantes. L'équation de Landau est utilisée pour modéliser des plasmas et prend la forme

$$\partial_t f + v \cdot \nabla_x f = Q_L(f, f).$$

Les collisions ici sont représentées par l'opérateur quadratique Q_L. Tout comme l'équation de Boltzmann avec collisions élastiques, l'équation de Landau conserve la masse, la quantité de mouvement et l'énergie et il existe un équivalent du Théorème H de Boltzmann pour l'équation de Landau.

Equations de Fokker-Planck

Parlons enfin d'une autre variante de l'équation de Boltzmann, l'équation de Fokker-Planck qui modélise un système de particules subissant un processus de diffusion et une force extérieure dérivant d'un potentiel. L'interaction entre ces deux processus est à la base de bon nombre de ses propriétés. L'équation de Fokker-Planck cinétique classique s'écrit sous la forme

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f + \mathrm{div}_v(f \, \nabla_v V),$$

avec par exemple $V = |v|^2/2$. Mentionnons également sa version homogène :

$$\partial_t f = \Delta_v f + \mathrm{div}_v(f \, \nabla_v V).$$

Contrairement aux modèles précédents, la seule quantité qui est conservée par cette équation est la masse. D'autres versions de cette équation peuvent être étudiées, notamment en remplaçant le laplacien par un terme diffusif plus général. Une généralisation intéressante est l'équation dans laquelle le laplacien est remplacé par un laplacien fractionnaire. Outre son intérêt propre, cette équation peut également être vue comme une caricature de l'équation linéarisée de Boltzmann sans troncature angulaire.

2 Problèmes mathématiques associés

Les deux grands problèmes qui sont abordés sur ces équations sont l'existence de solutions à ces équations et la stabilité de ces dernières. Nous présentons ici ces deux axes :

- **Problème A :** le problème de l'existence de solutions ;
- **Problème B :** le problème du comportement en temps grand des solutions.

A - Existence de solutions. Le problème de Cauchy est la première étape de l'étude mathématique d'équations cinétiques.

(A1) Le premier cadre qui a été considéré est le cadre des *solutions homogènes*, c'est-à-dire lorsque l'on suppose que la densité de particules f ne dépend pas de la position x. Nous n'entrerons pas ici dans les détails de cette théorie qui a largement été étudiée, mentionnons tout de même qu'elle a été introduite par Carleman pour une équation de Boltzmann avec sphères dures dans [22, 21].

(A2) Nous y reviendrons ultérieurement mais il est important ici de souligner que le théorème H de Boltzmann suggère que les solutions de l'équation convergent en temps grand vers un équilibre. Il apparait donc naturel de s'intéresser à des solutions qui sont proches de cet équilibre. Nous donnons ici quelques éléments de ce qui est appelée la théorie des *solutions perturbatives*. Sa base est la linéarisation de l'équation autour de l'équilibre et l'étude du problème de l'existence d'un trou spectral. L'étude de l'opérateur linéarisé remonte à Hilbert [54, 55], citons également les travaux fondateurs de Carleman [21] et Grad [48, 49]. Ce cadre a permis à Ukai [91] de construire les premières solutions globales de l'équation de Boltzmann et de donner le premier résultat de convergence exponentielle vers l'équilibre. Il a également permis de donner les premières justifications rigoureuses des limites hydrodynamiques.

(A3) Un autre cadre à l'étude du problème de Cauchy est celui des *solutions proches du vide* dans l'espace tout entier ($\Omega = \mathbb{R}^3$). Citons ici le premier travail de Illner et Shinbrot [56] pour l'équation de Boltzmann classique qui s'inscrit dans cette théorie ainsi que celui de Alonso [4] pour des collisions inélastiques.

(A4) Enfin, il est important d'évoquer la théorie des *solutions renormalisées* et l'article fondateur qui lui est associé de DiPerna et Lions [39]. Il s'agit d'une théorie qui a pu être appliquée à de nombreuses équations mais néanmoins pas au cas des collisions inélastiques du fait de la perte du Théorème H et du manque d'estimations de type entropie. Citons également le papier d'Alexandre et Villani [3] dans lequel est prouvée l'existence de solutions renormalisées à la DiPerna-Lions avec mesure de défaut pour le cas des équations de Boltzmann sans cut-off et Landau.

B - Comportement en temps grand. Le retour vers l'équilibre des solutions est une caractéristique qu'avait annoncée Boltzmann pour son équation, permettant ainsi de donner un cadre mathématique au second principe de la thermodynamique. Nous rappelons qu'une partie du théorème H de Boltzmann nous dit que l'entropie de Boltzmann décroit avec le temps (9). Il nous dit également que tout équilibre, c'est-à-dire toute distribution qui maximise l'entropie, est une distribution Maxwellienne. Il est donc attendu qu'une solution converge vers l'équilibre lorsque le temps t tend vers l'infini. Il s'agit donc de prouver cette convergence. Plusieurs techniques existent pour traiter cette question.

(B1) Une manière d'aborder le problème est d'utiliser des *méthodes d'entropie*. Il s'agit d'obtenir des inégalités impliquant la dissipation d'entropie $D(f)$ et l'entropie relative $H(f|\mu) := H(f) - H(\mu)$ (où μ désigne l'équilibre de l'équation) du type :

$$D(f) \geq K_f \, H(f|\mu)^\alpha.$$

Pour certaines équations, il peut être naturel de considérer d'autres fonctionnelles que la fonctionnelle H de Boltzmann, par exemple, on peut considérer des entropies relatives de la forme

$$\int \Phi\left(\frac{f}{\mu}\right) \mu - \Phi\left(\int f\right)$$

avec $\Phi(s) = s^p - 1 - p(s-1)$, $p \in (1,2]$, ou encore $\Phi(s) = s\,\log s - s + 1$. L'étude de l'entropie relative se fait alors grâce à des inégalités de type Poincaré ou Log-Sobolev.

Les inégalités obtenues sur l'entropie relative permettent d'obtenir des résultats de convergence vers l'équilibre en utilisant des inégalités de Csiszár-Kullback-Pinsker qui relient les normes L^p usuelles et les entropies.

(B2) Une deuxième technique est de *linéariser l'équation* étudiée (si elle n'est pas linéaire), ensuite d'étudier les propriétés spectrales de l'équation linéarisée (question de l'existence d'un trou spectral), puis de faire le lien, le cas échéant, entre l'équation linéarisée et l'équation non linéaire. Si l'équation est non linéaire, ceci donne une estimation de la vitesse de convergence dans un certain voisinage de l'équilibre, voisinage dans lequel les termes linéaires sont dominants sur les termes non linéaires de l'équation. Il peut donc être utile de combiner cette technique avec une autre pour savoir qu'à partir d'un certain temps, on sera précisément dans le bon voisinage dans lequel on peut utiliser la décroissance de la partie linéaire de l'équation.

Remarquons que ces techniques peuvent ne pas donner des taux explicites de convergence, notamment quand ils sont combinés avec des arguments de compacité. La méthode n'est alors pas constructive, ce qui implique que l'on n'obtienne pas d'information sur la vitesse de convergence.

3 Cadre général de la thèse

Dans cette thèse, des modèles homogènes en espace (équations de Fokker-Planck, de Boltzmann sans cut-off) et non homogènes (équations de Boltzmann inélastique, de Landau) sont étudiés. Une constante dans les stratégies mises en oeuvre dans l'étude de ces équations est la question de l'existence d'un trou spectral pour l'opérateur linéaire ou linéarisé dans divers espaces. Essentiellement deux arguments sont employés pour prouver l'existence de trous spectraux :

(C1) *Elargissement de l'espace dans lequel on a un trou spectral :* cette technique vaut pour des opérateurs dont on sait déjà qu'ils ont un trou spectral dans un “petit” espace.

(C2) *Argument perturbatif :* cette technique est utilisée lorsque l'opérateur considéré est vu comme une perturbation d'un opérateur limite dont on sait déjà qu'il admet un trou spectral.

De manière générale, dans le traitement d'un problème de Cauchy, les hypothèses naturelles qui sont faites sur la donnée initiale sont que la donnée initiale doit être de masse finie, d'énergie finie, éventuellement ayant des moments d'ordre plus grand finis ou encore d'entropie finie. Un cadre naturel dans lequel est développée une théorie de Cauchy est donc un cadre de type L^1 à poids polynomial. La finalité pour des équations qui sont déjà linéaires est tout simplement d'affaiblir les hypothèses nécessaires sur la donnée initiale pour avoir des hypothèses plus réalistes physiquement. Dans le cas où l'on étudie l'équation linéarisée d'une équation non linéaire, il est intéressant d'obtenir des résultats sur l'opérateur linéarisé dans des espaces de type L^1 à poids polynomial pour pouvoir faire le lien entre les théories linéarisée et non linéaire.

(C1) Elargissement de l'espace dans lequel l'opérateur linéaire a un trou spectral

Très souvent, des études sur l'existence d'un trou spectral ont déjà été menées dans un cadre Hilbert (typiquement un espace L^2 avec pour poids la racine de l'inverse de l'équilibre Maxwellien) dans lequel l'opérateur linéaire ou linéarisé est auto-adjoint. Le but est alors d'élargir l'espace dans lequel l'opérateur admet un trou spectral (stratégie mise en oeuvre dans les articles [90], [89], [28] et [71]). Il s'agit d'une méthode qui a été introduite par Mouhot dans [73] puis reprise et développée dans un cadre abstrait par Gualdani, Mischler et Mouhot dans [51]. On considère un opérateur linéaire $\mathcal{L}$ dont on sait déjà qu'il admet un trou spectral dans un "petit" espace E. L'opérateur $\mathcal{L}$ doit également pouvoir se décomposer en deux parties $\mathcal{L} = \mathcal{A} + \mathcal{B}$ avec $\mathcal{A}$ un opérateur borné, $\mathcal{B}$ un opérateur dissipatif et, $\mathcal{A}$ et $\mathcal{B}$ doivent être tels que le semi-groupe $\mathcal{A}S_{\mathcal{B}}(t)$ a des propriétés régularisantes (typiquement, s'il est convolé un nombre assez important de fois, il permet de passer du petit espace E à un plus grand espace $\mathcal{E}$). Sous ces hypothèses, l'opérateur $\mathcal{L}$ aura également un trou spectral dans $\mathcal{E}$.

(C2) Obtention de l'existence d'un trou spectral par un argument perturbatif

Cette méthode est employée lorsque l'opérateur linéaire ou linéarisé qui est étudié peut être vu comme une perturbation d'un opérateur limite dont on sait déjà qu'il a un trou spectral. Sous l'hypothèse que l'opérateur peut se décomposer d'une manière similaire à celle présentée au point précédent et en considérant de petites perturbations de l'opérateur limite, on en déduit de nouvelles propriétés spectrales sur l'opérateur perturbé.

On utilise cette méthode dans [88], ce qui nous donne de nouvelles propriétés spectrales pour notre opérateur linéarisé. Dans l'article [71], cette technique nous permet de faire le lien entre les propriétés spectrales de différentes équations et leurs équations limites de manière uniforme.

Etude des problèmes homogènes

Concernant l'étude des modèles homogènes, le problème de Cauchy a souvent déjà été traité. Nous nous concentrons donc sur la problématique du comportement en temps grand des solutions et également sur la vitesse du retour vers l'équilibre. Pour ce faire, comme annoncé précédemment, nous étudions systématiquement d'un point de vue spectral l'opérateur, linéaire si l'équation est linéaire (équations de Fokker-Planck [90, 71]), linéarisé autour de l'équilibre associé à l'équation si l'équation est non linéaire (équation de Boltzmann sans cut-off [89]). Plus précisément, ce qui nous intéresse est l'existence d'un *trou spectral* dans divers espaces, les plus larges possibles.

Pour l'équation de Fokker-Planck fractionnaire dans [90], grâce à cette technique, la classe des données initiales pour laquelle il y a un retour exponentiel vers l'équilibre a été largement élargie. Pour l'équation de Boltzmann sans cut-off dans [89], avec les nouvelles estimations spectrales obtenues dans des espaces L^1, nous parvenons à faire le lien avec la théorie non linéaire et obtenir un retour exponentiel vers l'équilibre des solutions du problème non linéaire. Enfin, dans [71], nous développons un cadre général dans lequel peuvent être traités les problèmes de retour vers l'équilibre pour différentes équations de Fokker-Planck de manière uniforme.

Etude des problèmes non homogènes

Pour les modèles non homogènes tels que celui de l'équation de Boltzmann inélastique [88] et celui de Landau [28], on traite conjointement le problème de Cauchy et le problème du retour vers l'équilibre. L'étude des problèmes linéaires mènent à de nouveaux résultats, on obtient l'existence d'un trou spectral dans de nouveaux espaces dans lesquels on peut établir une théorie de Cauchy et un retour exponentiel vers l'équilibre des solutions construites.

Espaces fonctionnels

Nous introduisons ici des notations concernant les espaces fonctionnels qui seront le cadre de notre étude, il s'agit de définir des espaces de Lebesgue et de Sobolev à poids. Nous allons considérer des fonctions de deux variables $h = h(x, v)$ avec $x \in \mathbb{T}^3$ et $v \in \mathbb{R}^3$ ou d'une seule variable $h = h(v)$ avec $v \in \mathbb{R}^3$. Soient $m = m(v)$ une mesure de Borel positive et et des réels $1 \le p, q \le \infty$. Lorsque $h = h(v)$ ne dépend que de v, on définit l'espace de Lebesgue à poids $L^p(m)$ comme l'espace associé à la norme

$$\|h\|_{L^p(m)} := \|hm\|_{L^p}.$$

On définit ensuite l'espace $L^p_v L^q_x(m)$ comme l'espace de Lebesgue associé à la norme, pour $h = h(x, v)$,

$$\begin{aligned} \|h\|_{L^p_v L^q_x(m)} &:= \big\| \|h\|_{L^q_x} m \big\|_{L^p_v} \\ &= \left(\int_{\mathbb{R}^3_v} \|h(\cdot, v)\|^p_{L^q_x} \, m(v)^p \, dv \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^3_v} \left(\int_{\mathbb{T}^3_x} |h(x,v)|^q \, dx \right)^{p/q} m(v)^p \, dv \right)^{1/p}. \end{aligned}$$

On définit également les espaces de Sobolev d'ordre supérieur pour $\ell \in \mathbb{N}$, lorsque $h = h(v)$, $W^{\ell,p}(m)$ sera l'espace associé à la norme

$$\|h\|_{W^{\ell,p}(m)} := \sum_{j=0}^{\ell} \|hm\|_{L^p}.$$

Enfin, on définit l'espace $W^{\ell,p}_v W^{n,q}_x(m)$, pour $\ell, n \in \mathbb{N}$ comme l'espace associé à la norme

$$\|h\|_{W^{\ell,p}_v W^{n,q}_x(m)} = \sum_{0 \le |\alpha| \le \ell, \, 0 \le |\beta| \le n, \, |\alpha| + |\beta| \le \max(\ell, n)} \big\| \, \|\partial^\alpha_v \partial^\beta_x h\|_{L^p_x} \, m(v) \, \big\|_{L^q_v}.$$

Lorsque $p = q$ et $\ell = n$, cette définition se réduit à celle d'un espace de Sobolev à poids usuelle $W^{\ell,p}_{x,v}(m)$ et nous noterons également $H^\ell = W^{\ell,2}$.

Nous en venons maintenant à la présentation détaillée des équations étudiées ainsi que des principaux résultats prouvés dans cette thèse.

4 Présentation des modèles étudiés

4.1 Equations cinétiques et collisions rasantes

Cette partie de la thèse s'articule autour de ce qu'on appelle les "collisions rasantes" c'est-à-dire les collisions entre particules pour lesquelles l'angle de déviation est proche de 0. En effet, nous étudions une équation de Boltzmann sans troncature angulaire, c'est-à-dire en tenant compte de l'effet des collisions rasantes. L'équation de Fokker-Planck fractionnaire peut elle être considérée comme un modèle simplifié de l'équation de Boltzmann sans cut-off linéarisée autour de l'équilibre. Quant à l'équation de Landau, il s'agit de considérer l'équation de Boltzmann dans la limite des collisions rasantes, c'est-à-dire rendre toutes les collisions rasantes.

Equation de Fokker-Planck fractionnaire

On se place ici dans un cadre homogène et la seule variable considérée est notée x. Pour $\alpha \in (0, 2)$, on considère la généralisation suivante de l'équation de Fokker-Planck :

$$\partial_t f = -\left(-\Delta\right)^{\alpha/2} f + \mathrm{div}_x(xf) =: \mathcal{L} f \quad \text{sur} \quad \mathbb{R}^d \tag{10}$$

avec donnée initiale $f_0 \geq 0$. L'opérateur $-(-\Delta)^{\alpha/2}$ est un laplacien fractionnaire et est défini de la manière suivante sur l'espace de Schwartz $\mathcal{S}(\mathbb{R}^d)$:

$$\left(-\Delta\right)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} \, dy$$

ou par sa transformée de Fourier

$$\mathcal{F}\left(\left(-\Delta\right)^{\alpha/2} f\right)(\xi) \approx |\xi|^\alpha \, \widehat{f}(\xi).$$

On peut élargir l'espace de fonctions sur lequel on définit le laplacien fractionnaire à

$$\left\{ f : \mathbb{R}^d \to \mathbb{R}, \ \int_{\mathbb{R}^d} \frac{|f(x)|}{1 + |x|^{d+\alpha}} \, dx < \infty \right\}$$

qui contient $\langle x \rangle^k$ pour $k \in (0, \alpha)$.

Les équations impliquant le lapacien fractionnaire ont tout d'abord un intérêt propre. Ce type d'opérateur apparait dans de nombreuses branches de la physique mais également en probabilités, le processus associé au laplacien fractionnaire et sa généralisation, les processus de Lévy ont des applications à la finance. Cette équation représente également un intérêt d'un point de vue technique car une partie de l'équation linéarisée de Boltzmann sans cut-off se comporte comme de la diffusion fractionnaire. Etudier cette équation constitue donc une première étape dans l'étude du modèle plus complexe de l'équation de Boltzmann sans cut-off qui est l'objet de la partie suivante.

Rappelons que si l'on considère l'équation de Fokker-Planck classique,

$$\partial_t f = \Delta f + \mathrm{div}_x(xf) \quad \text{sur } \mathbb{R}^d,$$

un laplacien apparait à la place du laplacien fractionnaire. Remarquons que si f est une fonction Schwartz, alors Δf l'est également. Ici, lorsque $\alpha \in (0,2)$, on voit apparaitre une singularité en 0 de la transformée de Fourier de $(-\Delta)^{\alpha/2} f$, ce qui induit un manque de décroissance à l'infini de $(-\Delta)^{\alpha/2} f$ qui n'est plus une fonction Schwartz mais qui, plus précisément, décroit à l'infini en $|x|^{-d-\alpha}$. Ceci induit un certain nombre de difficultés techniques dans la manipulation de certaines intégrales qui s'avèrent être moins flexibles.

Il a déjà été prouvé qu'il existe un unique équilibre de masse fixée pour l'équation (10), nous noterons μ l'unique équilibre de masse 1. De plus, on sait que cet équilibre est une distribution continue positive qui décroit à l'infini en $|x|^{-d-\alpha}$. Notons la différence, ici encore, avec l'équation de Fokker-Planck classique pour laquelle l'équilibre est une gaussienne.

De plus, grâce à des méthodes d'entropie, Gentil et Imbert ont prouvé dans [45] une convergence exponentielle vers l'équilibre de la solution de cette équation. Ils prouvent que si l'on considère une donnée initiale f_0 telle que $\|f_0 - \mu \langle f_0 \rangle\|_{L^2(\mu^{-1/2})}$ est finie alors il existe une constante explicite $\lambda_0 > 0$ telle que la solution f_t associée à f_0 satisfait

$$\forall\, t \geq 0, \quad \|f_t - \mu \langle f_0 \rangle\|_{L^2(\mu^{-1/2})} \leq e^{-\lambda_0 t} \|f_0 - \mu \langle f_0 \rangle\|_{L^2(\mu^{-1/2})}.$$

Le travail réalisé dans [90] consiste à prouver une convergence vers l'équilibre dans de plus grands espaces du type $L^1(\langle x \rangle^k)$ avec $k \in (0,\alpha)$, avec un taux exponentiel explicite de convergence et des hypothèses minimales sur la donnée initiale, à savoir $f_0 \in L^1(\langle x \rangle^k)$.

Voici le théorème principal démontré dans cet article :

Theorem 4.1. *Soit $k \in (0,\alpha)$. Pour toute donnée initiale $f_0 \in L^1(\langle x \rangle^k)$, la solution f_t de l'équation (10) satisfait la propriété de décroissance suivante :*

$$\forall\, t \geq 0, \quad \|f_t - \mu\langle f_0 \rangle\|_{L^1(\langle x \rangle^k)} \leq C\, e^{-\lambda t} \|f_0 - \mu\langle f_0 \rangle\|_{L^1(\langle x \rangle^k)}$$

où $\langle f_0 \rangle = \int_{\mathbb{R}^d} f_0$ et C, $\lambda > 0$ sont des constantes explicites.

La stratégie utilisée pour démontrer ce résultat est celle présentée dans le point **(C1)** de l'introduction avec les espaces $E := L^2(\mu^{-1/2})$ et $\mathcal{E} := L^1(\langle x \rangle^k)$ avec $k \in (0,\alpha)$.

Equation de Boltzmann sans cut-off

Nous considérons ici des particules décrites par leur densité homogène en espace $f = f(t, v)$. Nous étudions ainsi l'équation de Boltzmann dite homogène en espace :

$$\partial_t f = Q(f, f) \quad \text{sur} \quad \mathbb{R}^3. \tag{11}$$

L'opérateur de collision de Boltzmann Q est défini comme suit :

$$Q(g, f) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \left[g'_* f' - g_* f\right] \, d\sigma \, dv_*.$$

Dans ce qui précède et ce qui suit, nous utilisons les notations $f = f(v)$, $g_* = g(v_*)$, $f' = f(v')$ et $g'_* = g(v'_*)$. Dans ces expressions, v, v_* et v', v'_* sont les vitesses d'une paire de particules avant et après la collision. Nous considérons ici des collisions élastiques et nous rappelons que du fait des lois de conservation des collisions (3), les vitesses post-collision peuvent s'écrire sous la forme (4) :

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \, \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \, \sigma, \quad \sigma \in \mathbb{S}^2.$$

Le noyau de collision de Boltzmann $B(v - v_*, \sigma)$ dépend uniquement de la norme de la vitesse relative $|v - v_*|$ et de l'angle de déviation θ via $\cos\theta = \langle \kappa, \sigma \rangle$ où $\kappa = (v - v_*)/|v - v_*|$ et $\langle \cdot, \cdot \rangle$ est le produit scalaire usuel sur $\mathbb{R}^3$. Par un argument de symétrie, on peut toujours se ramener au cas où $B(v - v_*, \sigma)$ a son support inclus dans $\langle \kappa, \sigma \rangle \geq 0$ i.e. $0 \leq \theta \leq \pi/2$, hypothèse que nous faisons donc.

Dans ce travail, nous nous concentrons uniquement sur le cas où B satisfait les conditions qui suivent :

- il prend la forme suivante

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) \, b(\cos\theta); \tag{12}$$

- la fonction angulaire b est localement lisse, a une singularité non intégrable lorsque $\theta \to 0$ et satisfait pour un certain $c_b > 0$ et un certain $s \in (0, 1/2)$ (singularité angulaire modérée)

$$\forall \, \theta \in (0, \pi/2], \quad \frac{c_b}{\theta^{1+2s}} \leq \sin\theta \, b(\cos\theta) \leq \frac{1}{c_b \, \theta^{1+2s}}; \tag{13}$$

- le facteur cinétique Φ vérifie

$$\Phi(|v - v_*|) = |v - v_*|^\gamma \quad \text{avec} \quad \gamma \in (0, 1). \tag{14}$$

La principale motivation physique provient de particules qui interagissent selon un potentiel répulsif de la forme

$$\phi(r) = r^{-(p-1)}, \quad p \in (2, +\infty). \tag{15}$$

Les hypothèses qui sont faites sur B incluent le cas des potentiels de la forme (15) avec $p > 5$. En effet, pour des potentiels répulsifs de cette forme, le noyau de collision ne peut pas être calculé explicitement mais Maxwell a montré dans [62] qu'il peut être calculé en fonction du potentiel ϕ. Plus précisément, il satisfait les hypothèses précédentes (12), (13) et (14) en dimension 3 (voir [30, 31, 93]) avec $s := \frac{1}{p-1} \in (0,1)$ et $\gamma := \frac{p-5}{p-1} \in (-3,1)$.

L'usage est d'appeler *potentiels durs* le cas $p > 5$ (pour lequel $0 < \gamma < 1$), *molécules Maxwelliennes* le cas $p = 5$ (pour lequel $\gamma = 0$) et *potentiels mous* le cas $2 < p < 5$ (pour lequel $-3 < \gamma < 0$). On peut donc en déduire que les hypothèses faites sur B incluent le cas des potentiels durs.

Nous supposons dans ce travail que la donnée initiale f_0 est de masse 1, de moment nul et d'énergie 3, situation à laquelle on peut toujours se ramener. D'autre part, grâce aux propriétés de conservation de l'équation, une solution f_t de l'équation sera, en tout temps, de masse 1, de moment nul et d'énergie 3 :

$$\forall\, t \geq 0, \quad \int_{\mathbb{R}^3} f_t(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad i = 1, 2, 3.$$

Nous notons alors μ l'unique équilibre de même masse, moment et énergie que f_0 qui est donné par : $\mu(v) := (2\pi)^{-3/2} e^{-|v|^2/2}$.

Le problème de l'*existence de solutions* pour cette équation a déjà été traité. On sait que notre équation (11) admet des solutions qui sont conservatives et qui satisfont des propriétés de régularité, tous les moments et les normes Sobolev à poids polynomiaux sont produits instantanément. On appelle ce type de solutions "solutions lisses" et nous renvoyons au Chapitre 2 pour la définition précise. Le problème de Cauchy pour cette équation a d'abord été étudié par Arkeryd dans [10]. L'existence de solutions est prouvée pour des potentiels qui ne sont pas trop mous, en particulier pour les potentiels durs que nous considérons. L'unicité pour les potentiels durs peut être prouvée sous des conditions plus restrictives sur la donnée initiale (voir [37] et [43]).

L'objet de l'article [89] est l'étude du problème de *retour vers l'équilibre* des solutions lisses de cette équation. Ce problème a déjà été abordé par Carlen et Carvalho dans [23, 24] puis par Toscani et Villani dans [87]. Le meilleur taux de convergence obtenu jusqu'à présent a été obtenu par Villani dans [94], il s'agit d'un taux polynomial. Plus précisément, il montre que si f_t est une solution qui satisfait certaines propriétés de régularité de donnée initiale f_0 de masse, énergie et entropie finies, alors f_t vérifie : pour tout $t_0 > 0$ et tout $\varepsilon > 0$, il existe $C_{t_0,\varepsilon} > 0$ tel que

$$\forall\, t \geq t_0, \quad \|f_t - \mu\|_{L^1} \leq C_{t_0,\varepsilon}\, t^{-\frac{1}{\varepsilon}},$$

où on rappelle que μ est l'équilibre de l'équation.

Notre travail améliore ce résultat puisque nous prouvons une convergence exponentielle vers l'équilibre :

Theorem 4.2. *On considère un noyau de collision B qui satisfait les conditions (12), (13), (14) et f_0 une distribution positive de masse 1, de moment nul, d'énergie 3 et d'entropie finie. Si f_t est une solution "lisse" de l'équation (11) de donnée initiale f_0, alors*

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_{L^1} \leq C\, e^{-\lambda\, t}$$

où C et $\lambda > 0$ sont des constantes explicites.

La stratégie de la preuve est très différente de celle de la preuve du taux polynomial dans [94] dans la mesure où cette dernière est purement non linéaire. La technique que nous utilisons ici est la technique **(B2)** présentée dans le première partie de l'introduction. Revenons en détails ici sur la manière dont elle se décompose :

(a) Etude de l'équation linéarisée. On note $\mathcal{L}$ l'opérateur linéarisé défini par

$$\mathcal{L}h := Q(\mu, h) + Q(h, \mu).$$

Il est déjà connu que $\mathcal{L}$ a un trou spectral dans l'espace $L^2(\mu^{-1/2})$ [78, 58], la première estimation explicite ayant été obtenue par Baranger et Mouhot dans [15]. En utilisant la même technique que celle utilisée pour l'équation de Fokker-Planck fractionnaire, celle présentée au point **(C1)**, nous élargissons l'espace dans lequel nous avons une décroissance du semi-groupe. Plus précisément, nous obtenons une telle décroissance dans des espaces L^1 à poids polynomiaux. Si $k > 2$ est fixé, il existe des constantes explicites $C > 0$ et $\lambda > 0$ telles que pour tout $h \in L^1(\langle v\rangle^k)$,

$$\forall\, t \geq 0, \quad \|S_{\mathcal{L}}(t)h - \Pi h\|_{L^1(\langle v\rangle^k)} \leq C\, e^{-\lambda t}\|h - \Pi h\|_{L^1(\langle v\rangle^k)},$$

où $S_{\mathcal{L}}(t)$ est le semi-groupe associé à l'opérateur linéarisé $\mathcal{L}$ et Π est la projection sur l'espace propre associé à la valeur propre 0.

(b) Retour au problème non linéaire. La théorie de Cauchy pour cette équation est justement développée dans un espace L^1 à poids polynomial. Nous sommes donc en mesure de faire le lien entre les théories linéarisée et non linéaire. Nous utilisons le résultat de décroissance polynomiale vers l'équilibre de Villani pour savoir que notre solution atteindra tout voisinage de l'équilibre en un certain temps. Ce qui nous intéresse est de savoir qu'à partir d'un certain temps t_0, notre solution sera dans un voisinage convenable de l'équilibre dans lequel les termes linéaires de l'équation domineront les termes non linéaires. Ainsi, à partir de ce temps t_0, nous pourrons utiliser le résultat de décroissance exponentielle vers l'équilibre prouvé pour le semi-groupe de l'opérateur linéarisé.

Equation de Landau

Nous nous intéressons dans cette partie à l'équation de Landau inhomogène en espace qui décrit l'évolution d'une densité de particules $f = f(t, x, v)$. Dans le tore $\mathbb{T}^3$, l'équation est donnée par, pour $f = f(t, x, v) \geq 0$ avec $t \in \mathbb{R}^+$, $x \in \mathbb{T}^3$ et $v \in \mathbb{R}^3$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f, f) \\ f_{|t=0} = f_0. \end{cases} \tag{16}$$

L'opérateur de Landau Q est un opérateur bilinéaire qui prend la forme

$$Q(g,f)(v) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) \left[g_* \partial_j f - f \partial_j g_* \right] dv_*,$$

où nous utilisons la convention de sommation pour les indices répétés, et les dérivées sont des dérivées par rapport à la variable de vitesse v i.e $\partial_i = \partial_{v_i}$. Nous utilisons également les notations $g_* = g(v_*)$, $f = f(v)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $\partial_j f = \partial_{v_j} f(v)$, etc.

La matrice a_{ij} est symétrique définie positive et dépend du type d'interaction entre les particules que l'on considère. Elle est donnée par :

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3, 1].$$

Comme pour l'équation de Boltzmann, nous utilisons le vocabulaire usuel suivant : on parle de *potentiels durs* si $\gamma \in (0,1]$, de *molécules Maxwelliennes* si $\gamma = 0$, de *potentiels faiblement mous* si $\gamma \in [-2, 0)$, de *potentiels mous* si $\gamma \in (-3, -2)$ et de *potentiel Coulombien* si $\gamma = -3$. Dans le travail présenté dans cette thèse, seuls les potentiels durs, les molécules Maxwelliennes et les potentiels faiblement mous sont traités, c'est-à-dire les cas $\gamma \in [-2, 1]$.

Dans l'étude que nous réalisons, nous supposons que la donnée initiale f_0 est de masse 1, de moment nul et d'énergie 3, situation à laquelle on peut toujours se ramener. D'autre part, grâce aux propriétés de conservation de l'équation, une solution f de l'équation sera, en tout temps, de masse 1, de moment nul et d'énergie 3 :

$$\forall\, t \geq 0, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_t(x,v) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dx\, dv = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad i = 1, 2, 3.$$

Nous notons ainsi μ l'unique équilibre de même masse, moment et énergie que f_0 qui est donné par : $\mu(v) := (2\pi)^{-3/2} e^{-|v|^2/2}$.

Le problème de l'*existence de solutions* pour cette équation a déjà été abordé, pour des données initiales grandes, la notion de solutions renormalisées introduite par DiPerna-Lions pour l'équation de Boltzmann a été étendue à l'équation de Landau par Alexandre-Villani [3] où ils construisent des solutions renormalisées globales avec mesure de défaut. Puis, des solutions perturbatives ont été construites par Guo pour tous les cas $\gamma \in [-3, 1]$ dans des espaces de type $H^N_{x,v}(\mu^{-1/2})$ où μ désigne l'équilibre de l'équation avec $N \geq 8$. Ce résultat a été amélioré pour $\gamma \in [-2, 1]$ par Mouhot et Neumann dans [76] où ils prouvent le même type de résultat dans $H^N_{x,v}(\mu^{-1/2})$ pour $N \geq 4$.

Le résultat que nous prouvons dans [28] améliore largement les résultats précédents sur la théorie de Cauchy associée à l'équation de Landau dans la mesure où l'espace dans lequel nous travaillons est beaucoup moins restrictif. Tant au niveau du poids, nous pouvons travailler avec des poids polynomiaux ou stretched exponentiels au lieu de l'inverse de l'équilibre Maxwellien ; qu'au niveau des hypothèses sur les dérivées, en particulier, nous n'avons besoin d'aucune hypothèse sur les dérivées en vitesse.

De plus, nous étudions également le problème du *retour vers l'équilibre*. Nous citons ici les travaux de Guo et Strain [84, 85] qui prouvent un retour presque exponentiel vers l'équilibre pour les cas $\gamma \in [-3,-2)$, de Yu [103] et de Mouhot et Neumann [76] qui prouvent un retour exponentiel vers l'équilibre pour les cas $\gamma \in [-2,1]$ dans des espaces de type $H^N_{x,v}(\mu^{-1/2})$ avec respectivement $N \geq 8$ et $N \geq 1$. Ici encore, le résultat que nous obtenons dans [28] est meilleur puisque le retour exponentiel vers l'équilibre est prouvé dans des espaces plus larges.

Avant d'énoncer le résultat principal obtenu sur cette équation, nous apportons quelques précisions sur les notations utilisées.

Nous désignerons par m un poids qui vérifie l'une des hypothèses suivantes :

– pour les potentiels durs $\gamma \in (0,1]$ et molécules Maxwelliennes $\gamma = 0$:

  1. *poids polynomial* : $m = \langle v\rangle^k$ avec $k > \gamma + 7 + 3/2$,
  2. *poids stretched exponentiel* : $m = e^{r\langle v\rangle^s}$ avec $r > 0$ et $s \in (0,2)$,
  3. *poids exponentiel* : $m = e^{r\langle v\rangle^2}$ avec $r \in (0,1/2)$;

– pour les potentiels faiblement mous $\gamma \in [-2,0)$:

  1. *poids stretched exponentiel* : $m = e^{r\langle v\rangle^s}$ avec $r > 0$ et $s \in (-\gamma,2)$,
  2. *poids exponentiel* : $m = e^{r\langle v\rangle^2}$ avec $r \in (0,1/2)$.

Nous utiliserons les notations $\sigma = 0$ quand $m = \langle v\rangle^k$, et $\sigma = s$ quand $m = e^{r\langle v\rangle^s}$. L'espace $\mathcal{E}_0 := \mathcal{H}^3_x L^2_v(m)$ est l'espace associé à la norme

$$\begin{aligned}\|h\|^2_{\mathcal{H}^3_x L^2_v(m)} &:= \|h\|^2_{L^2_x L^2_v(m)} + \|\nabla_x h\|^2_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad + \|\nabla^2_x h\|^2_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} + \|\nabla^3_x h\|^2_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned}$$

On introduit aussi l'espace $H^1_{v,*}(m)$ associé à la norme anisotropique

$$\|h\|^2_{H^1_{v,*}(m)} := \|h\|^2_{L^2_v(m\langle v\rangle^{(\gamma+\sigma)/2})} + \|P_v \nabla_v h\|^2_{L^2_v(m\langle v\rangle^{\gamma/2})} + \|(I-P_v)\nabla_v h\|^2_{L^2_v(m\langle v\rangle^{(\gamma+2)/2})},$$

où P_v désigne la projection sur la droite engendrée par v, et enfin l'espace $\mathcal{E}_1 := \mathcal{H}^3_x H^1_{v,*}(m)$ associé à la norme

$$\begin{aligned}\|h\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} &:= \|h\|^2_{L^2_x H^1_{v,*}(m)} + \|\nabla_x h\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad + \|\nabla^2_x h\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} + \|\nabla^3_x h\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned}$$

Theorem 4.3. *Pour toute donnée initiale $f_0 \in \mathcal{E}_1$ de masse 1, de moment nul, d'énergie 3 et assez proche de μ dans l'espace $\mathcal{E}_1$, il existe une unique solution globale à l'équation de Landau (16), $f \in L^\infty_t([0,\infty);\mathcal{E}_0) \cap L^2_t([0,\infty);\mathcal{E}_1)$ qui satisfait de plus*

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_{\mathcal{E}_0} \leq C e^{-\lambda t} \|f_0 - \mu\|_{\mathcal{E}_0},$$

où C et $\lambda > 0$ sont des constantes explicites.

La stratégie de la preuve est différente des stratégies employées par Guo dans [52] et Mouhot-Neumann dans [76]. Guo utilise pour sa part une stratégie purement non linéaire, il utilise une méthode d'énergie et plus précisément, il obtient des estimations d'énergie directement sur le problème non linéaire. Mouhot et Neumann étudient eux l'opérateur linéarisé pour commencer. L'opérateur linéarisé non homogène en espace défini par

$$\mathcal{L}h := Q(h, \mu) + Q(\mu, h) - v \cdot \nabla_x h \tag{17}$$

peut être vu comme la somme de l'opérateur linéarisé homogène en espace et de l'opérateur de transport. Cela leur permet d'utiliser les propriétés de l'opérateur linéarisé homogène pour étudier l'opérateur linéarisé non homogène. Notre stratégie est celle présentée en **(A2)** et se décompose en deux étapes.

(a) **Etude de l'équation linéarisée.** Il s'agit d'étudier les propriétés spectrales de l'opérateur défini par (17). Il a déjà été prouvé que $\mathcal{L}$ admet un trou spectral dans des espaces de type $H^N_{x,v}(\mu^{-1/2})$ pour $N \geq 1$ par Mouhot et Neumann pour les cas $\gamma \in [-2, 1]$. On peut donc envisager d'appliquer la stratégie **(C1)** qui consiste à exhiber un découpage de $\mathcal{L}$ en deux parties qui vérifient certaines propriétés de régularité et de dissipativité. Nous élargissons alors l'espace dans lequel nous avons une décroissance exponentielle du semi-groupe. Nous obtenons en particulier l'existence de constantes explicites $C > 0$ et $\lambda > 0$ telles que pour tout $h \in \mathcal{E}_0$,

$$\forall\, t \geq 0, \quad \|S_{\mathcal{L}}(t)h - \Pi h\|_{\mathcal{E}_0} \leq C e^{-\lambda t} \|h - \Pi h\|_{\mathcal{E}_0},$$

où $S_{\mathcal{L}}(t)$ désigne le semi-groupe associé à $\mathcal{L}$ et Π la projection sur l'espace propre associé à la valeur propre 0 de $\mathcal{L}$.

(b) **Retour au problème non linéaire.** Nous obtenons tout d'abord des estimations a priori sur les solutions de l'équation non linéaire, estimations qui sont à la base de la mise en place d'un schéma itératif qui va nous permettre de construire une solution à l'équation. Pour que ces estimations a priori nous permettent d'avoir un schéma itératif stable et convergent, il faut que le gain qui provient de la partie linéaire de l'équation compense la perte induite par le terme bilinéaire de l'équation. Habituellement, des estimations naturelles sur le terme bilinéaire suffisent mais dans notre cas, nous avons besoin d'estimations très fines car le gain que l'on obtient avec la partie linéaire de l'équation est anisotropique. Grâce à ces estimations, nous sommes en mesure de prouver que, dans de bonnes normes, la partie linéaire de l'équation sera dominante sur la partie non linéaire, ce qui nous permet d'utiliser la décroissance exponentielle du semi-groupe de l'opérateur linéarisé obtenue précédemment.

4.2 Equation de Boltzmann avec collisions inélastiques

Equation de Boltzmann inélastique non homogène en espace avec terme de diffusion

Nous considérons dans cette partie des sphères dures décrites par leur densité $f = f(t, x, v)$. Nous nous plaçons dans un cadre où les particules entrent en collisions inélastiques dans le tore $\mathbb{T}^3$. La densité f satisfait l'équation de Boltzmann inélastique avec terme de diffusion suivante :

$$\partial_t f = Q_{e_\lambda}(f, f) + \lambda^\gamma \, \Delta_v f - v \cdot \nabla_x f. \tag{18}$$

Dans le cas d'un coefficient de restitution constant, $e_\lambda(\cdot)$ est constant égal à $1 - \lambda$ et γ est égal à 1. L'équation étudiée dans ce cas est donc

$$\partial_t f = Q_{1-\lambda}(f, f) + \lambda \, \Delta_v f - v \cdot \nabla_x f. \tag{19}$$

Nous renvoyons au Chapitre 4 pour expliquer d'où proviennent ces équations obtenues après changement d'échelle.

Concernant l'interprétation physique de cette équation, le terme $\lambda^\gamma \Delta_v f$ modélise l'effet d'un bain de chaleur constant, l'opérateur quadratique Q_{e_λ} modélise les collisions entre particules, l'inélasticité des collisions est caractérisée par le coefficient dit de restitution $e_\lambda(\cdot) := e(\lambda \, \cdot)$ avec e qui est soit constant soit qui satisfait les hypothèses suivantes :

– l'application $r \to e(r)$ de $\mathbb{R}^+$ dans $(0, 1]$ est décroissante et absolument continue ;
– l'application $r \to r\, e(r)$ est strictement croissante sur $\mathbb{R}^+$;
– il existe $a, b > 0$ et $\overline{\gamma} > \gamma > 0$ tels que

$$\forall \, r \geq 0, \quad |e(r) - 1 + a\, r^\gamma| \leq b\, r^{\overline{\gamma}}.$$

Ces hypothèses sont notamment vérifiées dans le cas de sphères dures viscoélastiques.

Définissons maintenant plus précisément l'opérateur de collision Q_{e_λ}. On note toujours v et v_* (respectivement v' et v'_*) les vitesses de deux particules avant (respectivement après) la collision. Rappelons qu'au cours d'une collision inélastique, la quantité de mouvement est préservée mais pas l'énergie, on peut donc écrire les lois physiques des collisions inélastiques comme suit :

$$v + v_* = v' + v'_*, \tag{20}$$

ce qui donne trois équations puisque nous sommes en dimension 3. Les vitesses v' et v'_* sont alors données par les formules suivantes

$$v' = v - \frac{1 + e_\lambda}{2} \, \frac{u - |u| \, \sigma}{2}, \quad v'_* = v_* + \frac{1 + e_\lambda}{2} \, \frac{u - |u| \, \sigma}{2}$$

où σ est un élément de la sphère $\mathbb{S}^2$ et $u := v - v_*$ est la vitesse relative. Cette représentation permet de donner une définition agréable de l'opérateur de collision sous forme faible :

$$\int_{\mathbb{R}^3} Q_{e_\lambda}(g, f) \, \psi \, dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} g(v_*) f(v) \left[\psi(v') \, - \, \psi(v)\right] |v - v_*| \, d\sigma \, dv_* \, dv,$$

pour toute fonction test $\psi = \psi(v)$ assez régulière. Remarquons qu'il y a bien dissipation d'énergie au cours de la collision :

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -|u|^2 \, \frac{1 - \widehat{u} \cdot \sigma}{4} \left(1 - e_\lambda \left(|u| \sqrt{\frac{1 - \widehat{u} \cdot \sigma}{2}} \right)^2 \right) \leq 0. \tag{21}$$

On remarque alors facilement que l'opérateur Q_{e_λ} conserve la masse et la quantité de mouvement, mais pas l'énergie grâce à (20) et (21). Il en est de même pour le laplacien qui préserve la masse et la quantité de mouvement mais qui a tendance à augmenter la température du gaz.

Nous considérons le cas où la donnée initiale f_0 de l'équation est de masse 1 et de moment nul, ce qui sera également le cas de toute solution au temps t de donnée initiale f_0 d'après les propriétés de conservation de la masse et du moment de l'équation :

$$\forall\, t \geq 0, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_t(v) \begin{pmatrix} 1 \\ v_i \end{pmatrix} dx\, dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

Précisons qu'il a déjà été démontré par Mischler et Mouhot pour un coefficient d'inélasticité constant et par Alonso et Lods pour le cas non constant dans [67, 8] que dans un régime de faible inélasticité, l'équation

$$\partial_t f = Q_{e_\lambda}(f, f) + \lambda^\gamma \Delta_v f$$

admet un unique équilibre de masse et moment fixés. Nous noterons G_λ l'unique équilibre de l'équation de masse 1 et de moment nul pour λ assez proche de 0 (c'est-à-dire pour une faible inélasticité). Insistons ici sur le fait que l'équilibre G_λ a une décroissance en l'infini de l'ordre de $e^{-|v|^{3/2}}$. Ces équilibres G_λ ne décroissent donc pas assez vite en l'infini pour appartenir à des espaces de type $L^2(e^{-|v|^2/2})$, espaces naturels pour étudier les propriétés spectrales de l'opérateur linéarisé élastique. Une théorie pertrubative autour du cas élastique n'est donc pas envisageable dans ce type d'espace. Mais grâce à leur théorie d'élargissement de l'espace de décroissance du semi-groupe, Gualdani, Mischler et Mouhot [51] ont élargi l'espace de décroissance du semi-groupe de l'opérateur linéarisé, permettant ainsi d'envisager une théorie perturbative autour de l'équation élastique.

L'objet de l'article [88] est l'étude du problème de l'*existence de solutions* pour les équations (18) et (19) et du *retour vers l'équilibre* de ces solutions. Le problème de Cauchy pour ces équations n'a été que peu abordé, ce qui s'explique notamment par le manque d'estimations de type entropie. Le théorème prouvé dans ce travail est le premier résultat d'existence de solutions perturbatives dans un "régime de collision", de plus, les deux cas des coefficients d'inélasticité constant et non constant sont traités. En effet, le seul résultat de ce type qui a été prouvé est celui d'Alonso dans un "régime proche du vide" [4].

Avant d'énoncer le résultat principal, il faut remarquer qu'il s'agit d'un résultat établi dans un régime de faible inélasticité. Il s'agit d'un choix qui s'explique mathématiquement mais aussi physiquement. Plus l'inélasticité est grande, plus il y a de liens créés entre les particules au cours des collisions binaires, l'hypothèse de chaos moléculaire de Boltzmann suggère donc de se placer dans un cadre de faible inélasticité. De plus, le champ d'application de cette hypothèse est large incluant notamment le cas des sphères dures viscoélastiques dont le coefficient de restitution est proche de 1 en moyenne. D'un point de vue plus mathématique, cette hypothèse de faible inélasticité permet de considérer notre équation comme une petite perturbation autour de l'équation élastique, ce qui nous permet donc d'utiliser les résultats déjà connus sur cette dernière. Enonçons maintenant le principal résultat de [88] :

Theorem 4.4. *Soit* $\mathcal{E}_0 = W_x^{s,1} W_v^{2,1}\left(\langle v\rangle e^{b\langle v\rangle^\beta}\right)$ *où* $b > 0$*,* $\beta \in (0,1)$ *et* $s > 6$*. Pour* λ *assez petit, et pour toute donnée initiale* $f_0 \in \mathcal{E}_0$ *de masse* 1*, de moment nul et assez proche de l'équilibre* G_λ *dans* $\mathcal{E}_0$*, il existe une unique solution globale* $f \in L_t^\infty(\mathcal{E}_0)$ *à l'équation (18) qui satisfait de plus*

$$\forall\, t \geq 0, \quad \|f_t - G_\lambda\|_{\mathcal{E}_0} \leq C\, e^{-\alpha\, t}\, \|f_0 - G_\lambda\|_{\mathcal{E}_0}$$

où C *et* $\alpha > 0$ *sont des constantes explicites.*

Dans le cas d'un coefficient d'inélasticité constant, la conclusion du théorème reste vraie si l'on prend une donnée initiale $f_0 \in \mathcal{E}_0$ *assez proche d'une distribution homogène en espace* $g_0 = g_0(v)$ *dans* $\mathcal{E}_0$*.*

La stratégie de la preuve est celle présentée au point **(A2)** que l'on détaille dans notre cas plus précisément. Elle se décompose comme précédemment en deux temps.

(a) Etude de l'équation linéarisée. On utilise ici la méthode exposée au point **(C2)**. On note $\mathcal{L}_\lambda$ l'opérateur linéarisé défini par

$$\mathcal{L}_\lambda h := Q_{e_\lambda}(G_\lambda, h) + Q_{e_\lambda}(h, G_\lambda) + \lambda^\gamma\, \Delta_v h - v \cdot \nabla_x h.$$

Le point essentiel de cette étude est de se placer dans un régime de faible inélasticité et de considérer l'opérateur linéarisé inélastique comme une perturbation de l'opérateur linéarisé élastique. Ceci, dans le but d'utiliser les connaissances sur l'opérateur linéarisé élastique pour en déduire des propriétés sur l'opérateur linéarisé inélastique. Grâce aux récents développements sur les propriétés spectrales de l'opérateur linéarisé élastique, nous sommes en mesure de prouver de nouvelles estimations spectrales. Plus précisément, nous montrons que si $\lambda > 0$ est assez petit, il existe $\alpha_\lambda > 0$ tel que pour tout $h \in \mathcal{E}_0$:

$$\forall\, t \geq 0, \quad \|S_{\mathcal{L}_\lambda}(t)h - \Pi S_{\mathcal{L}_\lambda}(t)h\|_{\mathcal{E}_0} \leq C e^{-\alpha_\lambda t} \|f - \Pi h\|_{\mathcal{E}_0}$$

où $S_{\mathcal{L}_\lambda}(t)$ est le semi-groupe associé à l'opérateur linéarisé $\mathcal{L}_\lambda$ et Π est la projection sur l'espace propre associé à la valeur propre 0.

(b) Retour au problème non linéaire. En combinant ces nouvelles estimations spectrales sur l'opérateur linéarisé avec des estimations sur le terme bilinéaire de l'équation, nous construisons des solutions perturbatives autour de l'équilibre. Nous avons exploité des estimations bilinéaires déjà connues dans le cas homogène en espace pour les étendre au cas non homogène. La construction de solutions se fait grâce à l'obtention d'estimations a priori puis de la mise en place d'un schéma itératif dont la convergence est prouvée grâce aux estimations a priori, le point essentiel étant que la perte dû au terme bilinéaire de l'équation est compensée par le gain obtenu grâce à la partie linéaire. La partie linéaire est donc dominante sur la partie non linéaire, ce qui nous permet d'utiliser la décroissance exponentielle obtenue sur le semi-groupe associé à l'opérateur linéarisé.

4.3 Equations de Fokker-Planck

Analyse spectrale uniforme des équations de Fokker-Planck discrète, fractionnaire et classique

Dans cette partie, nous nous intéressons à l'analyse spectrale des équations de Fokker-Planck discrète, fractionnaire et classique du point de vue des semi-groupes. Nous sommes dans un cadre homogène et la seule variable considérée est notée x. Ce sont des modèles simples qui permettent de décrire l'évolution d'une densité de particules $f = f(t,x)$ soumise à de la diffusion et à un mécanisme de confinement qui s'écrivent sous la forme

$$\partial_t f = \mathcal{D}_\varepsilon f + \mathrm{div}_x(xf) =: \Lambda_\varepsilon f. \tag{22}$$

Le terme de diffusion peut être soit discret :

$$\mathcal{D}_\varepsilon f := \frac{1}{\varepsilon^2}(k_\varepsilon * f - f)$$

pour un noyau k convenable (à symétrie radiale, positif, régulier et suffisamment décroissant) et avec la notation usuelle $k_\varepsilon(x) = 1/\varepsilon^d k(x/\varepsilon)$, $x \in \mathbb{R}^d$. Il peut également être un terme de diffusion fractionnaire

$$\mathcal{D}_\varepsilon(f)(x) := c_\varepsilon \int_{\mathbb{R}^d} \frac{f(y) - f(x) - \chi(x-y)(y-x)\cdot\nabla f(x)}{|x-y|^{d+2-\varepsilon}}\, dy,$$

avec $\varepsilon \in (0,2)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbb{1}_{B(0,1)} \le \chi \le \mathbb{1}_{B(0,2)}$, et une constante de renormalisation convenable $c_\varepsilon > 0$. Les deux familles d'équations sont reliées à l'équation de Fokker-Planck classique car à la limite $\varepsilon \to 0$, on retrouve l'équation de Fokker-Planck classique :

$$\partial_t f = \frac{1}{2}\Delta f + \mathrm{div}(xf) =: \Lambda_0 f.$$

Les principales caractéristiques de ces équations sont les suivantes : la masse est conservée par l'équation ainsi que la positivité et elles admettent un unique état d'équilibre noté G_ε de masse 1 qui est exponentiellement stable, en particulier

$$f(t) \xrightarrow[t\to\infty]{} G_\varepsilon \tag{23}$$

pour toute solution f associée à une donnée initiale masse nulle. Ces résultats peuvent être obtenus de différentes manières, par exemple en utilisant des inégalités de type Poincaré ou des inégalités de Sobolev logarithmiques, ou encore en utilisant la théorie de Krein-Rutman sur les semi-groupes positifs.

Dans l'article [71], le travail qui est réalisé permet de traiter dans un même cadre toutes ces équations et également d'obtenir le fait que la convergence (23) est exponentiellement rapide de manière uniforme par rapport à ε pour une large classe de données initiales. Le résultat principal que nous obtenons peut s'écrire de la manière suivante :

Theorem 4.5. *Pour tout* $\varepsilon_0 \in (0,2)$*, il existe* $a < 0$ *et* $C \geq 1$ *tels que pour tout* $f \in X$ *et tout* $\varepsilon \in [0, \varepsilon_0]$*,*

$$\|S_{\Lambda_\varepsilon}(t) f - G_\varepsilon \langle f \rangle\|_X \leq C\, e^{at}\, \|f\|_X, \quad \forall\, t \geq 0,$$

où $\langle f \rangle = \int_{\mathbb{R}^d} f$*.*

Selon le modèle considéré, X peut être un espace de Lebesgue à poids polynomial $L^p(\langle x \rangle^q)$ ou encore un espace de Sobolev à poids polynomial $W^{s,p}(\langle x \rangle^q)$.

D'une part, ce résultat généralise les résultats obtenus sur l'équation de Fokker-Planck classique dans [51, 64] au cas discret et l'analyse qui est faite est réalisée de manière uniforme par rapport au paramètre de "discrétisation". D'autre part, ce résultat rend uniforme par rapport au paramètre de diffusion fractionnaire les résultats obtenus sur l'équation de Fokker-Planck fractionnaire dans [90].

L'existence d'un trou spectral puis d'une décroissance exponentielle vers l'équilibre pour les équations (22) peut être établie grâce à la théorie de Krein-Rutman développée dans [70]. Néanmoins, cette technique ne donne pas un trou spectral et un taux de décroissance uniformes en ε. Pour obtenir cette uniformité, nous utilisons principalement les deux techniques présentées **(C1)** et **(C2)**. Concernant l'utilisation de **(C1)**, nous l'utilisons pour le passage de Fokker-Planck fractionnaire à Fokker-Planck classique. Il est déjà connu que notre équation admet un trou spectral dans $L^2(G_\varepsilon^{-1/2})$ qui est uniforme en ε. Nous prouvons ensuite que les propriétés vérifiées par le découpage de l'opérateur $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ sont uniformes en ε, ce qui nous permet d'obtenir un trou spectral indépendant de ε dans $L^1(\langle x \rangle^k)$. L'utilisation de la technique **(C2)** repose sur les connaissances que l'on a sur l'équation limite, l'équation de Fokker-Planck classique. Ainsi, en prouvant que notre opérateur Λ_ε est une petite perturbation de l'opérateur limite Λ_0, nous obtenons des informations sur le comportement de Λ_ε uniformément en ε.

5 Liste de travaux rassemblés dans la thèse

Les chapitres de ce manuscrit sont composés des travaux suivants :

- Chapitre 1 : article [90], à paraitre dans *Communication in Mathematical Sciences.*
- Chapitre 2 : article [89], paru dans *Journal of statistical physics.*
- Chapitre 3 : article [28], écrit en collaboration avec Kleber Carrapatoso et Kung-Chien Wu, *prépublication.*
- Chapitre 4 : article [88], *prépublication.*
- Chapitre 5 : article [71], écrit en collaboration avec Stéphane Mischler, *en préparation.*

6 Perspectives

Les problèmes qu'il semble naturel d'envisager après cette thèse sont présentés brièvement dans ce qui suit.

1. **Estimations de convergence dans des limites singulières.**
   – Un premier problème à étudier serait le traitement dans un cadre uniforme des équations de Fokker-Planck discrète et fractionnaire, tout comme ce qu'il a été fait dans [71] pour les équations de Fokker-Planck discrète et classique. Les difficultés techniques sont plus importantes du fait du laplacien fractionnaire qui nous donnent des contraintes fortes sur le poids de l'espace considéré pour pouvoir mener les calculs.
   – Il s'agirait également de généraliser les travaux effectués dans [71] dans plusieurs directions : étudier le même type d'équation que (22) avec un noyau k plus singulier, typiquement en considérant $k = 1/2(\delta_1 + \delta_{-1})$ en dimension 1 ; étudier le même type d'équation avec un confinement plus général que celui utilisé en incluant d'autres forces ou un terme de confinement discret.
   – Une autre possibilité qui s'inscrit dans la suite du travail [71] est de faire une étude semblable pour l'équation de Fokker-Planck cinétique en utilisant des idées développées dans [27] qui ont permis d'unifier d'un point de vue spectral les équations de Keller-Segel parabolique-parabolique et parabolique-elliptique.
   – Enfin, le développement d'un cadre général pour l'analyse spectrale des semi-groupes de diverses équations de Fokker-Planck dans [71] pourrait être adapté à des équations non linéaires. On peut s'attendre à pouvoir traiter dans un même cadre uniforme les équations de Landau, Boltzmann sans cut-off et Boltzmann avec cut-off, ce qui permettrait d'obtenir une convergence exponentielle vers l'équilibre Maxwellien de manière uniforme pour ces équations.

2. **Equation de Boltzmann sans cut-off homogène en espace.** L'étude réalisée dans l'article [89] est faite dans un cadre de potentiels durs. Il serait naturel de considérer le même problème dans le cas des potentiels faiblement mous, d'autant que le problème a été résolu pour l'équation de Landau avec potentiels faiblement mous dans [26]. De plus, l'étude de l'équation linéarisée a été uniquement réalisée dans un cadre L^1, il serait intéressant de le généraliser à d'autre espaces L^p, en vue d'étudier notamment le problème non homogène en espace.

3. **Equation de Boltzmann sans cut-off non homogène en espace.** L'étude, d'une part, de l'équation de Boltzmann sans cut-off dans un cadre homogène en espace [89] et d'autre part, le développement d'une théorie de Cauchy pour l'équation de Landau non homogène en espace [28] sont deux éléments solides pour envisager d'étudier le problème de Cauchy pour l'équation de Boltzmann sans cut-off non homogène en espace. La stratégie d'approche serait similaire à celle utilisée pour l'équation de Landau.

4. **Limite hydrodynamique de l'équation de Boltzmann inélastique.** Le développement d'une théorie de Cauchy pour l'équation de Boltzmann inélastique avec

terme diffusif dans [88] rend envisageable l'étude des limites macroscopiques de l'équation en se basant sur l'étude du problème linéarisé qui a été réalisée.

Première partie

Equations cinétiques et collisions rasantes

Chapitre 1

Fractional Fokker-Planck equation

Résumé. Cette partie traite du comportement à long terme des solutions de l'équation de “Fokker-Planck fractionnaire” qui est de la forme $\partial_t f = I[f] + \mathrm{div}(xf)$ où l'opérateur I est un laplacien fractionnaire. Nous prouvons une décroissance exponentielle des solutions vers l'équilibre dans de nouveaux espaces. En effet, un tel résultat a déjà été obtenu dans un espace L^2 avec un poids prescrit par l'équilibre dans [45]. Nous améliorons ce résultat en obtenant une telle décroissance dans un espace L^1 à poids polynomial. Pour ce faire, nous tirons parti du récent article [51] dans lequel une théorie abstraite d'élargissement de l'espace fonctionnel de décroissance du semi-groupe est développée.

Abstract. This part deals with the long time behavior of solutions to a “fractional Fokker-Planck" equation of the form $\partial_t f = I[f] + \mathrm{div}(xf)$ where the operator I stands for a fractional Laplacian. We prove an exponential in time convergence towards equilibrium in new spaces. Indeed, such a result was already obtained in a L^2 space with a weight prescribed by the equilibrium in [45]. We improve this result obtaining the convergence in a L^1 space with a polynomial weight. To do that, we take advantage of the recent paper [51] in which an abstract theory of enlargement of the functional space of the semigroup decay is developed.

1.1 Introduction

1.1.1 Model and main result

For $\alpha \in (0,2)$, we consider the following generalization of the Fokker-Planck equation:

$$\partial_t f = -(-\Delta)^{\alpha/2} f + \mathrm{div}(xf), \quad \text{in } \mathbb{R}^d \tag{1.1}$$

with an initial data f_0. In the sequel, we will use the shorthand notations

$$I[f] = -(-\Delta)^{\alpha/2} f \quad \text{and} \quad \mathcal{L}f = I[f] + \mathrm{div}(xf).$$

The operator $(-\Delta)^{\alpha/2}$ is a fractional Laplacian, we first define it on the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ and we then extend the definition to others functions. We refer to Section 1.2 for the exact definition and for properties.

We also define here weighted L^p spaces in the following way: for some given Borel weight function $m \geq 0$ on $\mathbb{R}^d$, let us define $L^p(m)$, $1 \leq p \leq +\infty$, as the Lebesgue space associated to the norm

$$\|h\|_{L^p(m)} = \|h\, m\|_{L^p}.$$

Finally, we introduce the notation $\langle x \rangle := (1+|x|^2)^{1/2}$ for any $x \in \mathbb{R}^d$.

Before going into the statement of our main result, we here mention that it is a known fact that there exists a unique steady state of (1.1) of mass 1 and we denote it μ (see Subsection 1.4.1 for more details).

Theorem 1.1.6. *Let us consider $k \in (0,\alpha)$. For any $a \in (-\min(\lambda,k),0)$ (where $\lambda > 0$ will be defined in Corollary 1.4.16) and for any initial data $f_0 \in L^1(\langle x \rangle^k)$, the solution $f(t)$ of the equation (1.1) satisfies the following decay:*

$$\forall\, t \geq 0, \quad \|f(t) - \mu\langle f_0 \rangle\|_{L^1(\langle x \rangle^k)} \leq C_a\, e^{at}\, \|f_0 - \mu\langle f_0\rangle\|_{L^1(\langle x \rangle^k)}$$

where $\langle f_0 \rangle = \int_{\mathbb{R}^d} f_0$ and for some constant $C_a > 0$.

1.1.2 Known results

The main references to mention here are the papers [16] and [45]. In these two papers, “Lévy-Fokker-Planck equations" (the fractional Laplacian is replaced by a Lévy operator) are studied using the entropy production method. There is a proof of existence and uniqueness of a nonnegative steady state of mass 1 of the associated stationary equation. Then, in a weighted L^2 space with a weight prescribed by the equilibrium, a convergence (with an exponential rate) of the solution of the full equation towards equilibrium is obtained. Let us give more details about these results. We first introduce the main tools used.

Consider a smooth convex function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ and ν positive of mass 1 and define the Φ-entropy: for any nonnegative function f,

$$\mathrm{Ent}_\nu^\Phi(f) := \int_{\mathbb{R}^d} \Phi(f)\, \nu\, dx - \Phi\left(\int_{\mathbb{R}^d} f\, \nu\, dx\right).$$

Jensen's inequality gives that $\mathrm{Ent}_\nu^\Phi(f) \geq 0$. Let f_0 be an initial condition of a Lévy-Fokker-Planck equation or of the classical Fokker-Planck equation:

$$\partial_t f = \Delta f + \mathrm{div}(xf), \quad \text{in } \mathbb{R}^d \tag{1.2}$$

Then, let us introduce the quantity $E_\Phi(f_0)(t) := \mathrm{Ent}_\mu^\Phi\left(\frac{f(t)}{\mu}\right)$ (where μ denote the unique steady state of the equation considered of mass 1) which is well-defined for any $t > 0$.

In the case of the classical Fokker-Planck equation (1.2), by using functional inequalities as Poincaré, logarithmic Sobolev or Φ-entropy inequalities, one obtains exponential decays to zero of $E_\Phi(f_0)$. Then, the solution f of (1.2) converges towards the steady state of mass 1 in the sense of Φ-entropy. Methods to prove such results are usually based on entropy/entropy-production tools. See [9, 12, 13, 32] for different methods and applications.

In [16], Biler and Karch study Lévy-Fokker-Planck equations where the Lévy operators are Fourier multipliers associated to symbols $a(\xi)$ satisfying for some real number $\beta \in (0, 2]$

$$0 < \liminf_{\xi \to 0} \frac{a(\xi)}{|\xi|^\beta} \leq \limsup_{\xi \to 0} \frac{a(\xi)}{|\xi|^\beta} < \infty \quad \text{and} \quad 0 < \inf \frac{a(\xi)}{|\xi|^2}.$$

They prove that there exist $C > 0$ and $\varepsilon > 0$ such that

$$E_{|\cdot|^2/2}(f_0)(t) \leq Ce^{-\varepsilon t},$$

which means that the solution converges towards equilibrium at an exponential rate in $L^2(\mu^{-1/2})$. They deduce a similar result in L^2 and finally, under some more restrictive regularity and decay assumptions on f_0, they prove that the exponential convergence holds in L^1.

In [45], taking advantage of the paper [16], Gentil and Imbert prove an exponential decay of the Φ-entropies for a class of convex functions Φ and for a larger class of operators which includes the fractional Laplacian.

In the present paper, we only consider the equation (1.1) but we are able to enlarge the space where we have a decay towards equilibrium with minimal assumptions on f_0. If we compare our result to the one obtained in [16] for others operators defined above, we have to underline the fact that the result of convergence of the solution towards equilibrium in L^1 from [16] requires additional assumptions on f_0 (f_0 must have finite moments of a large order), it is not the case in our main result where f_0 is only supposed to belong to $L^1(\langle x \rangle^k)$ with $k < \alpha$.

1.1.3 Method of proof and outline of the paper

The main outcome of the present paper is a result of decay towards equilibrium with an exponential rate of convergence in $L^1(\langle x \rangle^k)$ (with $k < \alpha$) for solutions of our equation (1.1). To do that, we adopt the same strategy as the one adopted in [51] by Gualdani, Mischler and Mouhot for the classical Fokker-Planck equation. Let us explain in more details this strategy. It is based on the theory of enlargement of the functional space of the semigroup decay developed in [51]. It enables to get a spectral gap in a larger space when we already have one in a smaller space. It applies to operators $\mathcal{L}$ which can be splitted into two parts, $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A}$ bounded and $\mathcal{B}$ dissipative. Moreover, if we denote $e^{\mathcal{B}t}$ the semigroup associated to the operator $\mathcal{B}$, the semigroup $\left(\mathcal{A}\, e^{\mathcal{B}t}\right)$ is required to have some regularization properties. The fact that we can use this theory for our operator is based on two facts:

- we know from [45] that our operator has a spectral gap in $L^2(\mu^{-1/2})$ where we recall that μ is the only steady state of mass 1 of (1.1),
- we are able to get a splitting satisfying the previous properties using computations based on properties of the fractional Laplacian.

In section 1.2, we recall some technical tools about the fractional Laplacian that are useful in order to get a splitting of the operator. In section 1.4, we state results from [45] which are necessary to apply the abstract theorem of enlargement of spectral gap, which is reminded in Section 1.3. Finally, in Section 1.5, we apply this theorem to obtain our main result on the convergence towards equilibrium of the solution of (1.1) in $L^1(\langle x\rangle^k)$ with $k < \alpha$.

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1.2 Preliminaries on the fractional Laplacian

In this section, we recall some elementary properties of the fractional Laplacian that we will need through this paper. The usual reference for this kind of operators is Landkof's book [60].

1.2.1 Definition on $\mathcal{S}\left(\mathbb{R}^d\right)$

Let us consider $\alpha \in (0,2)$. The fractional Laplacian $(-\Delta)^{\alpha/2}$ is an operator defined on $\mathcal{S}(\mathbb{R}^d)$ by:

$$\forall\, f \in \mathcal{S}(\mathbb{R}^d), \quad (-\Delta)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}}\, dy. \tag{1.3}$$

This definition has to be understood in the sense of principal value:

$$(-\Delta)^{\alpha/2} f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}}\, dy.$$

Due to the singularity of the kernel, the right hand-side of (1.3) is not well defined in general. However, when $\alpha \in (0,1)$, the integral is not really singular near x. Indeed, since $f \in \mathcal{S}(\mathbb{R}^d)$, both f and ∇f are bounded. We hence deduce the following inequality:

$$\left|\int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}}\, dy\right| \leq \|\, \nabla f\|_{L^\infty} \int_{\overline{B}(x,1)} \frac{dy}{|x-y|^{d+\alpha-1}} + \|f\|_{L^\infty} \int_{\mathbb{R}^d\setminus\overline{B}(x,1)} \frac{dy}{|x-y|^{d+\alpha}}.$$

When $\alpha \in (0,2)$, we can also write the fractional Laplacian with a non principal value integral. For any $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\forall\, x \in \mathbb{R}^d, \quad (-\Delta)^{\alpha/2} f(x) = -\frac{1}{2}\int_{\mathbb{R}^d} \frac{f(x+y)+f(x-y)-2f(x)}{|y|^{d+\alpha}}\, dy \tag{1.4}$$

and this integral is well defined.

We can extend the integral definition of the fractional Laplacian to the following set of functions:

$$\left\{ f : \mathbb{R}^d \to \mathbb{R},\ \int_{\mathbb{R}^d} \frac{|f(x)|}{1+|x|^{d+\alpha}}\, dx < \infty \right\}$$

In particular, we can define $(-\Delta)^{\alpha/2}\langle x\rangle^k$ when $k < \alpha$.

1.2.2 Fractional Laplacian and Fourier transform

Let us remind a well-known fact about the Fourier transform of the fractional Laplacian of a Schwartz function.

Lemma 1.2.7. *There exists $C > 0$ such that for any $f \in \mathcal{S}\left(\mathbb{R}^d\right)$, we have:*

$$\mathcal{F}\left((-\Delta)^{\alpha/2} f\right)(\xi) = C\,|\xi|^{\alpha}\, \widehat{f}(\xi).$$

If f is a Schwartz function, there is a singularity at 0 in the Fourier transform of $(-\Delta)^{\alpha/2} f$. It implies a lack of decay at infinity for $(-\Delta)^{\alpha/2} f$ itself, $(-\Delta)^{\alpha/2} f$ is not a Schwartz function. We can prove that $(-\Delta)^{\alpha/2} f$ decays at infinity as $|x|^{-d-\alpha}$.

We now mention a very useful property of the fractional Laplacian which can be seen as a sort of integration by parts.

Lemma 1.2.8. *Let us consider f and g two Schwartz functions. Then, we have*

$$\int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} f(x)\, g(x)\, dx = \int_{\mathbb{R}^d} f(x)\, (-\Delta)^{\alpha/2} g(x)\, dx.$$

If $k < \alpha$, we can also prove that

$$\int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} f(x)\, \langle x\rangle^k\, dx = \int_{\mathbb{R}^d} f(x)\, (-\Delta)^{\alpha/2} \langle x\rangle^k\, dx.$$

1.2.3 Fractional Laplacian and fractional Sobolev spaces

Most of the time, fractional Sobolev spaces $H^s\left(\mathbb{R}^d\right)$ are defined in the following way: $H^s\left(\mathbb{R}^d\right) = W^{s,2}\left(\mathbb{R}^d\right)$ for $s \geq 0$ is the set of functions $f \in L^2\left(\mathbb{R}^d\right)$ such that $\left[\left(1+|\cdot|^2\right)^{s/2} \widehat{f}\right]$ is also in $L^2\left(\mathbb{R}^d\right)$. We remind here an equivalent definition which is going to be useful in what follows.

Lemma 1.2.9. *Let us consider $s \in (0,1)$. We have:*

$$H^s\left(\mathbb{R}^d\right) = \left\{ f \in L^2\left(\mathbb{R}^d\right) \, : \, \frac{|f(x)-f(y)|}{|x-y|^{\frac{d}{2}+s}} \in L^2\left(\mathbb{R}^d \times \mathbb{R}^d\right) \right\}.$$

We also have the following fact:

$$\left\| (-\Delta)^{\alpha/4} f \right\|^2_{L^2\left(\mathbb{R}^d\right)} = C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+\alpha}} \, dy \, dx$$

for some $C > 0$.

1.3 Theorem of enlargement of the functional space of the semigroup decay

1.3.1 Notations

For a given real number $a \in \mathbb{R}$, we define the half complex plane

$$\Delta_a := \{ z \in \mathbb{C}, \, \Re e \, z > a \}.$$

For some given Banach spaces $(E, \| \cdot \|_E)$ and $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$ we denote by $\mathscr{B}(E, \mathcal{E})$ the space of bounded linear operators from E to $\mathcal{E}$ and we denote by $\| \cdot \|_{\mathscr{B}(E,\mathcal{E})}$ or $\| \cdot \|_{E \to \mathcal{E}}$ the associated norm operator. We write $\mathscr{B}(E) = \mathscr{B}(E,E)$ when $E = \mathcal{E}$. We denote by $\mathscr{C}(E, \mathcal{E})$ the space of closed unbounded linear operators from E to $\mathcal{E}$ with dense domain, and $\mathscr{C}(E) = \mathscr{C}(E,E)$ in the case $E = \mathcal{E}$.

For a Banach space X and $\Lambda \in \mathscr{C}(X)$ we denote by $e^{\Lambda t}$, $t \geq 0$, its semigroup, by $\mathrm{D}(\Lambda)$ its domain, by $\mathrm{N}(\Lambda)$ its null space and by $\mathrm{R}(\Lambda)$ its range. We also denote by $\Sigma(\Lambda)$ its spectrum, so that for any z belonging to the resolvent set $\rho(\Lambda) := \mathbb{C} \backslash \Sigma(\Lambda)$ the operator $\Lambda - z$ is invertible and the resolvent operator

$$\mathcal{R}_\Lambda(z) := (\Lambda - z)^{-1}$$

is well-defined, belongs to $\mathscr{B}(X)$ and has range equal to $\mathrm{D}(\Lambda)$. We recall that $\xi \in \Sigma(\Lambda)$ is said to be an eigenvalue if $\mathrm{N}(\Lambda - \xi) \neq \{0\}$. Moreover, an eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be isolated if

$$\Sigma(\Lambda) \cap \{ z \in \mathbb{C}, \; |z - \xi| \leq r \} = \{\xi\} \; \text{ for some } r > 0.$$

In the case when ξ is an isolated eigenvalue, we may define $\Pi_{\Lambda,\xi} \in \mathscr{B}(X)$ the associated spectral projector by

$$\Pi_{\Lambda,\eta} := -\frac{1}{2i\pi} \int_{|z-\xi|=r'} (\Lambda - z)^{-1} \, dz$$

with $0 < r' < r$. Note that this definition is independent of the value of r' as the application $\mathbb{C} \setminus \Sigma(\Lambda) \to \mathscr{B}(X)$, $z \to \mathcal{R}_\Lambda(z)$ is holomorphic. For any $\xi \in \Sigma(\Lambda)$ isolated, it is well-known (see [57] paragraph III-6.19) that $\Pi_{\Lambda,\xi}^2 = \Pi_{\Lambda,\xi}$, so that $\Pi_{\Lambda,\xi}$ is indeed a projector.

When moreover the so-called "algebraic eigenspace" $\mathrm{R}(\Pi_{\Lambda,\xi})$ is finite dimensional we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$. In that case, $\mathcal{R}_\Lambda$ is a meromorphic function on a neighborhood of ξ, with non-removable finite-order pole ξ.

Finally for any $a \in \mathbb{R}$ such that

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$$

where $\xi_1, \dots, \xi_k$ are distinct discrete eigenvalues, we define without any risk of ambiguity

$$\Pi_{\Lambda,a} := \Pi_{\Lambda,\xi_1} + \dots \Pi_{\Lambda,\xi_k}.$$

We shall also need the following definition on the convolution of semigroups. Consider some Banach spaces X_1, X_2 and X_3. For two given functions

$$S_1 \in L^1\left(\mathbb{R}^+; \mathscr{B}(X_1, X_2)\right) \quad \text{and} \quad S_2 \in L^1\left(\mathbb{R}^+; \mathscr{B}(X_2, X_3)\right),$$

the convolution $S_2 * S_1 \in L^1\left(\mathbb{R}^+; \mathscr{B}(X_1, X_3)\right)$ is defined by

$$\forall\, t \geq 0, \quad S_2 * S_1(t) = \int_0^t S_2(s)\, S_1(t-s)\, ds.$$

When $S_1 = S_2$ and $X_1 = X_2 = X_3$, $S^{(*\ell)}$ is defined recursively by $S^{(*1)} = S$ and $S^{(*\ell)} = S^{(*(\ell-1))}$ for any $\ell \geq 2$.

Let us now introduce the notion of hypodissipative operators (we refer to [51, Subsection 2.3] for further details on this subject). If one consider a Banach space $(X, \|\cdot\|_X)$ and some operator $\Lambda \in \mathscr{C}(X)$, $(\Lambda - a)$ is said to be hypodissipative on X if there exists some norm $|||\cdot|||_X$ on X equivalent to the initial norm $\|\cdot\|_X$ such that

$$\forall\, f \in \mathrm{D}(\Lambda), \quad \exists\, \phi \in F(f) \quad \text{s.t} \quad \Re e\langle \phi, (\Lambda - a) f\rangle \leq 0, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket for the duality in X and X^* and $F(f) \subset X^*$ is the dual set of f defined by

$$F(f) = F_{|||\cdot|||_X}(f) := \left\{ \phi \in X^*,\ \langle \phi, f \rangle = |||f|||_X^2 = |||\phi|||_{X^*}^2 \right\}.$$

We also mention that if Λ is a generator of a semigroup $e^{\Lambda t}$, the fact that $(\Lambda - a)$ is hypodissipative on X is equivalent to the existence of a constant $C \geq 1$ such that the semigroup $e^{\Lambda t}$ satisfies

$$\forall\, t \geq 0, \quad \|e^{\Lambda t}\|_{\mathscr{B}(X)} \leq C\, e^{at}. \tag{1.6}$$

Moreover, when $|||\cdot|||_X$ is an Hilbert norm on X, we have $F(f) = \{f\}$ and (1.5) writes

$$\forall\, f \in \mathrm{D}(\Lambda), \quad \Re e\, (\!(f, (\Lambda - a) f)\!)_X \leq 0$$

where $(\!(\cdot, \cdot)\!)_X$ is the scalar product associated to $|||\cdot|||_X$. Finally, we notice that a dissipative operator is nothing but an hypodissipative one satisfying the previous definition with $|||\cdot|||_X = \|\cdot\|_X$ or, equivalently, satisfying the semigroup estimate (1.6) with $C = 1$.

1.3.2 The abstract theorem

Let us now present an enlargement of the functional space of a quantitative spectral mapping theorem (in the sense of semigroup decay estimate). The aim is to enlarge the space where the decay estimate on the semigroup holds. The version stated here comes from [51, Theorem 2.13] and [51, Lemma 2.17].

Theorem 1.3.10. *Let E, $\mathcal{E}$ be two Banach spaces such that $E \subset \mathcal{E}$ with dense and continuous embedding, and consider $L \in \mathscr{C}(E)$, $\mathcal{L} \in \mathscr{C}(\mathcal{E})$ with $\mathcal{L}_{|E} = L$ and $a \in \mathbb{R}$. We assume:*

(1) *L generates a semigroup e^{tL} and*

$$\Sigma(L) \cap \Delta_a = \{\xi_1, \ldots, \xi_k\} \subset \Sigma_d(L)$$

(with $\xi_k \neq \xi_{k'}$ if $k \neq k'$ and $\{\xi_1, \ldots, \xi_k\} = \emptyset$ if $k = 0$) and $L - a$ is dissipative on $\mathrm{R}\,(\mathrm{Id} - \Pi_{L,a})$.

(2) *There exist $\mathcal{A}$, $\mathcal{B} \in \mathscr{C}(\mathcal{E})$ such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ (with corresponding restrictions A and B on E) and some constants $\ell_0 \in \mathbb{N}^*$, $C \geq 1$, $b \in \mathbb{R}$ and $\gamma \in [0, 1)$ so that*

- **(i)** *$B - a$ and $\mathcal{B} - a$ are hypodissipative respectively on E and $\mathcal{E}$,*
- **(ii)** *$A \in \mathscr{B}(E)$ and $\mathcal{A} \in \mathscr{B}(\mathcal{E})$,*
- **(iii)** *$T_{\ell_0} := \left(\mathcal{A}\, e^{\mathcal{B}t}\right)^{(*\ell_0)}$ satisfies*

$$\forall\, t \geq 0, \quad \|T_{\ell_0}(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq C\, \frac{e^{bt}}{t^\gamma}.$$

Then the following estimate on the semigroup holds:

$$\forall\, a' > a,\, \forall\, t \geq 0, \quad \left\| e^{\mathcal{L}t} - \sum_{j=1}^{k} e^{Lt}\, \Pi_{\mathcal{L},\xi_j} \right\|_{\mathscr{B}(\mathcal{E})} \leq C_{a'}\, e^{a't}.$$

Remark 1.3.11. *The assumption (2)-(iii) implies that for any $a' > a$, there exist some constructive constants $n \in \mathbb{N}$, $C_{a'} \geq 1$ such that*

$$\forall\, t \geq 0, \quad \|T_n(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq C_{a'}\, e^{a't}.$$

1.4 Semigroup decay in $L^2(\mu^{-1/2})$ where μ is the steady state

1.4.1 Preliminaries on steady states

We recall results obtained in [45] about existence of steady states. They prove such a theorem for a more general equation than ours:

$$\begin{aligned} \partial_t f &= \mathcal{I}[f] + \mathrm{div}\,(f\, \nabla V) \quad && x \in \mathbb{R}^d,\, t > 0 \\ f(0,x) &= f_0(x) && x \in \mathbb{R}^d \end{aligned}$$

where $f_0 \in L^1\left(\mathbb{R}^d\right)$. The operator $\mathcal{I}$ is a Lévy operator defined as:

$$\mathcal{I}[f](x) = \operatorname{div}(\sigma\nabla f)(x) - b\cdot\nabla f(x) + \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \nabla f(x)\cdot z\, h(z)\right)\nu(dz)$$

where σ is a symmetric semi-definite $d\times d$ matrix, $b\in\mathbb{R}^d$, ν denotes a nonnegative singular measure on $\mathbb{R}^d$ that satisfies $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d}\min\left(1,|z|^2\right)\nu(dz) < \infty$ and h is a truncature function, $h(z) = 1/(1+|z|^2)$ for example.

The fractional Laplacian corresponds to a particular Lévy operator. Indeed, with $\sigma = 0$, $b = 0$ and $\nu(dz) = |z|^{-d-\alpha}\,dz$, we obtain the fractional Laplacian. In this particular case, the proof of existence of steady states of (1.1) is easier, we hence give a sketch of a proof of it (it is adapted from the proof of [45, Theorem 1]).

We suppose that μ is an equilibrium of the equation (1.1). At least formally, we have:

$$I[\mu] + \operatorname{div}(x\mu) = 0. \tag{1.7}$$

We do the following computation in order to take the Fourier transform of (1.7)

$$\begin{aligned}\mathcal{F}(\operatorname{div}(x\mu))(\xi) &= \sum_{j=1}^d \mathcal{F}(\partial_j(x_j\mu))(\xi) = \sum_{j=1}^d i\,\xi_j\mathcal{F}(x_j\mu)(\xi)\\ &= -\sum_{j=1}^d \xi_j\partial_j\widehat{\mu}(\xi) = -\xi\cdot\nabla\widehat{\mu}(\xi).\end{aligned}$$

We deduce that an equilibrium μ satisfies

$$|\xi|^\alpha\,\widehat{\mu}(\xi) + \xi\cdot\nabla\widehat{\mu}(\xi) = 0,$$

which implies that $\widehat{\mu}(\xi) = Ce^{-|\xi|^\alpha/\alpha}$ for some constant $C > 0$ and thus $\mu = C\,\mathcal{F}^{-1}\left(e^{-|\cdot|^\alpha/\alpha}\right)$. The constant C can be chosen so that $\int_{\mathbb{R}^d}\mu = 1$.

As announced in the introduction, we denote μ the only steady state of (1.1) of mass 1.

Remark 1.4.12. *Let us make some comments about this steady state μ.*

1. *It is a continuous positive distribution (cf [82] and [79, 86] for the positivity).*
2. *Concerning the behavior at infinity, in the case of the classical Fokker-Planck equation (1.2), the steady state is a Maxwellian, it is hence a Schwartz function. In our case, the steady state is not anymore a Schwartz function because its Fourier transform has a singularity at 0. If we denote χ_1 a smooth function which is nonnegative, supported on $|x|\le 2$ and such that $\chi_1(x) = 1$ for $|x|\le 1$, we can write the following decomposition of $\widehat{\mu}$:*

   $$\widehat{\mu}(\xi) = \chi_1(\xi)\left(1 + a_1|\xi|^\alpha + a_2|\xi|^{2\alpha} + \dots\right) + (1-\chi_1(\xi))\,e^{-|\xi|^\alpha/\alpha}.$$

   We see that the second part of the right-hand side is a Scwhartz function and the first one induces a singularity at 0. We can hence prove that

   $$\mu(x) \approx |x|^{-d-\alpha} \quad \textit{when} \quad |x| \to \infty.$$

1.4.2 Decay properties in $L^2(\mu^{-1/2})$

We again use results obtained in [45]. We just use them in our particular case, the fractional Laplacian.

For Φ a convex function, we introduce D_Φ on $\left(\mathbb{R}^+\right)^2$ as:

$$D_\Phi(a,b) = \Phi(a) - \Phi(b) - \Phi'(b)(a-b),$$

which is nonnegative on $\left(\mathbb{R}^+\right)^2$.

We will not prove the next two lemmas which are going to enable us to prove the decay towards equilibrium in $L^2(\mu^{-1/2})$. The first one is [45, Proposition 1] and the second one is [45, Theorem 2] and comes from [32].

Lemma 1.4.13. *Consider f_0 a nonnegative initial data for the equation (1.1) which satisfies $Ent_\mu^\Phi\left(\frac{f_0}{\mu}\right) < \infty$. Then, for any smooth convex function Φ and for any $t \geq 0$, the solution $f(t)$ satisfies*

$$\frac{d}{dt} E_\Phi\left(f_0\right)(t) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D_\Phi\left(u(t,x), u(t,x-z)\right)\frac{dz}{|z|^{d+\alpha}}\,\mu(x)\,dx$$

where $u(t,x) = f(t,x)/\mu(x)$.

Lemma 1.4.14. *Let us suppose that Φ is a smooth convex function such that*

$$(a,b) \mapsto D_\Phi(a+b,b) \text{ is convex on } \{a+b \geq 0, b \geq 0\}. \tag{1.8}$$

Then, for any smooth function v, we have:

$$Ent_\mu^\Phi\left(v(t,\cdot)\right) \leq K\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D_\Phi(v(t,x), v(t,x+z))\frac{dz}{|z|^{d+\alpha}}\,\mu(x)\,dx$$

for some $K > 0$.

We can now state the main theorem ([45, Theorem 1]) of this section, its proof is a direct consequence of the two previous lemmas and the Gronwall lemma.

Theorem 1.4.15. *Consider Φ a smooth convex function satisfying (1.8) and a nonnegative initial data f_0 such that $Ent_\mu^\Phi\left(\frac{f_0}{\mu}\right) < \infty$. Then, the estimate on the solution $f(t)$ holds:*

$$\forall\, t \geq 0, \quad Ent_\mu^\Phi\left(\frac{f(t)}{\mu}\right) \leq e^{-t/K}\, Ent_\mu^\Phi\left(\frac{f_0}{\mu}\right).$$

Corollary 1.4.16. *Consider a nonnegative initial data f_0 such that $\|f_0 - \mu\langle f_0\rangle\|_{L^2\left(\mu^{-1/2}\right)}$ is finite. Then there exists $\lambda > 0$ such that the solution $f(t)$ satisfies*

$$\forall\, t \geq 0, \quad \|f(t) - \mu\langle f_0\rangle\|_{L^2\left(\mu^{-1/2}\right)} \leq e^{-\lambda t}\,\|f_0 - \mu\langle f_0\rangle\|_{L^2\left(\mu^{-1/2}\right)}.$$

Proof. Theorem 1.4.15 applied with $\Phi(s) = (s - \langle f_0\rangle)^2$ gives the result. □

Remark 1.4.17. *Since $\|\cdot\|_{L^2(\mu^{-1/2})}$ is an Hilbert norm, using the previous corollary, we have for any $f \in L^2(\mu^{-1/2})$,*

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}(f - \mu\langle f\rangle)\,(f - \mu\langle f\rangle)\,\mu^{-1} &= \int_{\mathbb{R}^d} \mathcal{L}f\, f\,\mu^{-1} \\ &\le -\lambda\,\|f - \mu\langle f\rangle\|^2_{L^2(\mu^{-1/2})}.\end{aligned}$$

In what follows, we denote $\widetilde{\mu}(x) := \langle x\rangle^{-d-\alpha}$. We now give a corollary which gives a decay property in the space $L^2(\widetilde{\mu}^{-1/2})$ i.e $L^2\left(\langle x\rangle^{(d+\alpha)/2}\right)$.

Corollary 1.4.18. *Consider a nonnegative initial data f_0 such that $\|f_0 - \mu\,\langle f_0\rangle\|_{L^2\left(\widetilde{\mu}^{-1/2}\right)}$ is finite. Then, there exists $C > 0$ such that:*

$$\|f(t) - \mu\,\langle f_0\rangle\|_{L^2\left(\widetilde{\mu}^{-1/2}\right)} \le C\,e^{-\lambda t}\,\|f_0 - \mu\,\langle f_0\rangle\|_{L^2\left(\widetilde{\mu}^{-1/2}\right)},$$

where λ is defined in Corollary 1.4.16.

Proof. We use Remark 1.4.12 which implies that there exist two constants C_1, $C_2 > 0$ such that $C_1\,\mu \le \widetilde{\mu} \le C_2\,\mu$. As a consequence, we have the following series of inequalities:

$$\begin{aligned}\|f(t) - \mu\,\langle f_0\rangle\|_{L^2\left(\widetilde{\mu}^{-1/2}\right)} &\le C_1^{-1}\,\|f(t) - \mu\,\langle f_0\rangle\|_{L^2\left(\mu^{-1/2}\right)} \\ &\le C_1^{-1}\,e^{-\lambda t}\,\|f_0 - \mu\,\langle f_0\rangle\|_{L^2\left(\mu^{-1/2}\right)} \\ &\le C_2\,C_1^{-1}\,e^{-\lambda t}\,\|f_0 - \mu\,\langle f_0\rangle\|_{L^2\left(\widetilde{\mu}^{-1/2}\right)},\end{aligned}$$

which concludes the proof. □

1.5 Semigroup decay in $L^1(\langle x\rangle^k)$

1.5.1 Splitting of the operator

We would like to get a splitting of our operator $\mathcal{L}$ into two operators which satisfies hypothesis of Theorem 1.3.10 with $E = L^2(\widetilde{\mu}^{-1/2})$ and $\mathcal{E} = L^1(\langle x\rangle^k)$ with $k < \alpha$. In what follows, we denote $m(x) := \langle x\rangle^k$, $k < \alpha$.

Lemma 1.5.19. *Consider $a \in (-\min(k,\lambda), 0)$ where $\lambda > 0$ is defined in Corollary 1.4.16. There exist two operators $\mathcal{A}$ and $\mathcal{B}$ which satisfy the following conditions:*

(i) $\mathcal{L} = \mathcal{A} + \mathcal{B}$,

(ii) $\mathcal{A} \in \mathscr{B}(L^2(\widetilde{\mu}^{-1/2}))$ *and* $\mathcal{A} \in \mathscr{B}(L^1(m))$,

(iii) $\mathcal{B} - a$ *is hypodissipative on* $L^2(\widetilde{\mu}^{-1/2})$ *and* $L^1(m)$.

Proof. We are going to estimate the integral $\int \mathcal{L}f \,\mathrm{sign}(f)\, m$ with f a Schwartz function. The inequality obtained will also hold for any $f \in L^1(m)$ because of the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^1(m)$. We split the integral into two parts:

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}f \,\mathrm{sign}(f)\, m &= \int_{\mathbb{R}^d} I[f] \,\mathrm{sign}(f)\, m + \int_{\mathbb{R}^d} \mathrm{div}(xf) \,\mathrm{sign}(f)\, m \\ &=: T_1 + T_2.\end{aligned}$$

As far as T_1 is concerned, we introduce the function $\Phi(s) := |s|$ on $\mathbb{R}^d$ which is convex and its derivative is $\Phi'(s) = \mathrm{sign}(s)$. We also introduce the notation $K(x) := |x|^{-d-\alpha}$. Let us do the following computation:

$$\begin{aligned}&\int_{\mathbb{R}^d} (f(y) - f(x))\, K(x-y)\, dy \;\mathrm{sign}(f(x)) \\ &= \int_{\mathbb{R}^d} \left((f(y) - f(x))\Phi'(f(x)) + \Phi(f(x)) - \Phi(f(y))\right) \, K(x-y)\, dy \\ &\quad + \int_{\mathbb{R}^d} \left(\Phi(f(y)) - \Phi(f(x))\right) \, K(x-y)\, dy \\ &\le \int_{\mathbb{R}^d} \left(|f|(y) - |f|(x)\right) \, K(x-y)\, dy = I[|f|](x),\end{aligned}$$

where the last inequality comes from the convexity of Φ. We hence deduce that

$$T_1 \le \int_{\mathbb{R}^d} \; I[|f|]\, m = \int_{\mathbb{R}^d} |f|\, I[m] = \int_{\mathbb{R}^d} |f|\, m\, \frac{I[m]}{m},$$

because of Lemma 1.2.8.

Let us now deal with T_2. Performing integrations by parts, we obtain:

$$\begin{aligned}T_2 &= \int_{\mathbb{R}^d} \mathrm{div}(xf) \,\mathrm{sign}(f)\, m \\ &= d \int_{\mathbb{R}^d} \; |f|\, m + \int_{\mathbb{R}^d} x \cdot \nabla f \,\mathrm{sign} f\, m \\ &= d \int_{\mathbb{R}^d} \; |f|\, m + \int_{\mathbb{R}^d} x \cdot \nabla |f|\, m \\ &= d \int_{\mathbb{R}^d} \; |f|\, m - d \int_{\mathbb{R}^d} \; |f|\, m - \int_{\mathbb{R}^d} |f|\, x \cdot \nabla m \\ &= - \int_{\mathbb{R}^d} |f|\, m\, \frac{x \cdot \nabla m}{m}.\end{aligned}$$

We now introduce $\psi_{m,1} := I\,[m]\,/m - x \cdot \nabla m/m$. Let us study the behavior of $\psi_{m,1}$ at infinity. First, $x \cdot \nabla m(x)/m(x)$ tends to k as $|x|$ tends to infinity. Then, we prove that $I[m](x)/m(x)$ tends to 0 as $|x|$ tends to infinity. We use both representations (1.3) and (1.4) to split $I[m](x)$ into two parts:

$$I[m](x) = \frac{1}{2}\int_{|z|\ \le 1} (m(x+z)+m(x-z)-2m(x))\ K(z)\,dz$$
$$+\int_{|x-y|\ \ge 1} (m(y)-m(x))\ K(x-y)\,dy$$
$$=: I_1[m](x) + I_2[m](x).$$

Concerning $I_1[m]$, using a Taylor expansion, we obtain:

$$\begin{aligned} |m(x+z)+m(x-z)-2m(x)| &\le \sup_{|z|\le 1} \|D^2 m(x+z)\|_\infty\, |z|^2 \\ &\le C\langle x\rangle^{k-2}|z|^2, \end{aligned}$$

from which we deduce that

$$I_1[m](x) \le C\langle x\rangle^{k-2}\int_{|z|\le 1} \frac{dz}{|z|^{d+\alpha-2}}. \tag{1.9}$$

Concerning $I_2[m]$, let us introduce the function $\psi(s) := s^{k/2}$ on $\mathbb{R}^+$. Using the fact that ψ is $k/2$-Hölder continuous on $\mathbb{R}^+$ because $k/2 \le 1$, we obtain for any x, $y \in \mathbb{R}^d$:

$$\left|\psi(1+|x|^2) - \psi(1+|y|^2)\right| \ \le C\left||x|^2-|y|^2\right|^{k/2}$$

for some $C > 0$. We deduce the following inequalities:

$$\begin{aligned} |m(x)-m(y)| &\le C\,||x|-|y||^{k/2}(|x|+|y|)^{k/2} \\ &\le C\,|x-y|^{k/2}(|x|+|x-y|+|x|)^{k/2} \\ &\le C\left(|x-y|^{k/2}|x|^{k/2}+|x-y|^k\right). \end{aligned}$$

Finally, we obtain the following estimate on $I_2[m]$:

$$I_2[m](x) \le C\left(|x|^{k/2}\int_{|z|\ge 1}\frac{dz}{|z|^{d+\alpha-k/2}} + \int_{|z|\ge 1}\frac{dz}{|z|^{d+\alpha-k}}\right), \tag{1.10}$$

where we notice that the integrals are convergent because $k < \alpha$.

Gathering (1.9) and (1.10), we deduce that $I[m]/m$ tends to 0 at infinity. Finally, we obtain:

$$\int_{\mathbb{R}^d} \mathcal{L}f\,\mathrm{sign}f\, m \le \int_{\mathbb{R}^d} |f|\, m\,\psi_{m,1} \quad \text{with} \quad \lim_{|x|\to\infty}\psi_{m,1}(x) = -k < 0.$$

We introduce the smooth function χ_R ($R > 0$) which is nonnegative, supported on $|x| \le 2R$ and such that $\chi_R(x) = 1$ for $|x| \le R$. For any $a > -\min(\lambda, k)$, we may find M and R large enough so that

$$\forall\, x \in \mathbb{R}^d, \quad \psi_{m,1}(x) - M\chi_R(x) \le a. \tag{1.11}$$

Indeed, if we choose R large enough such that for any $|x| \geq R$, $\psi_{m,1}(x) \leq a$ and $M := \max_{|x|\leq R} \psi_{m,1}(x) - a$, we have (1.11).

We then introduce $\mathcal{A} := M\chi_R$ and $\mathcal{B} := \mathcal{L} - M\chi_R$. We finally obtain:

$$\begin{aligned}\int_{\mathbb{R}^d} (\mathcal{B} - a)\, f\, \mathrm{sign} f\, m &= \int_{\mathbb{R}^d} (\mathcal{L} - M\chi_R - a)\, f\, \mathrm{sign} f\, m \\ &\leq \int_{\mathbb{R}^d} (\psi_{m,1} - M\chi_R - a)\, |f|\;\, m \\ &\leq 0,\end{aligned}$$

which implies that $\mathcal{B} - a$ is dissipative on $L^1(m)$.

Let us now check that $\mathcal{B} - a$ is hypodissipative on $L^2(\widetilde{\mu}^{-1/2})$, using Remark 1.4.17, we obtain:

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{B}f\, f\, \mu^{-1} &= \int_{\mathbb{R}^d} \mathcal{L}f\, f\, \mu^{-1} - M \int_{\mathbb{R}^d} \chi_R\, f^2 \mu^{-1} \\ &\leq -\lambda\, \|f - \mu\langle f\rangle\|^2_{L^2(\mu^{-1/2})} - M \int_{\mathbb{R}^d} \chi_R\, f^2 \mu^{-1} \qquad (1.12) \\ &= -\lambda \left(\|f\|^2_{L^2(\mu^{-1/2})} - \langle f\rangle^2 \right) - M \int_{\mathbb{R}^d} \chi_R\, f^2 \mu^{-1}\end{aligned}$$

Using Cauchy-Schwarz inequality, we have:

$$\langle f\rangle^2 \leq C_0 \int_{\mathbb{R}^d} f^2\, \langle x\rangle^{d+\frac{\alpha}{2}},$$

for some constant $C_0 > 0$. We now use Remark 1.4.12. On the one hand, possibly increasing the value of R, we can suppose that

$$|x| \geq R \Rightarrow \langle x\rangle^{d+\frac{\alpha}{2}} \leq \frac{a+\lambda}{\lambda\, C_0}\, \mu^{-1},$$

which implies that

$$\lambda\, C_0 \int_{|x|\geq R} f^2\, \langle x\rangle^{d+\frac{\alpha}{2}} \leq (a+\lambda) \int_{|x|\geq R} f^2\, \mu^{-1}. \qquad (1.13)$$

On the other hand, up to increase the value of M, we can suppose that

$$|x| \leq R \Rightarrow \langle x\rangle^{d+\frac{\alpha}{2}} \leq \frac{M}{\lambda\, C_0}\, \mu^{-1},$$

we deduce that

$$\lambda\, C_0 \int_{|x|\leq R} f^2\, \langle x\rangle^{d+\frac{\alpha}{2}} \leq M \int_{|x|\leq R} f^2\, \mu^{-1}. \qquad (1.14)$$

Gathering (1.13) and (1.14), we can conclude that

$$\lambda\, \langle f\rangle^2 \leq (a+\lambda) \int_{\mathbb{R}^d} f^2\, \mu^{-1} + M \int_{|x|\leq R} f^2\, \mu^{-1}.$$

Going back to (1.12), we finally obtain

$$\int_{\mathbb{R}^d} \mathcal{B}f\, f\, \mu^{-1} \le a\, \|f\|^2_{L^2(\mu^{-1/2})},$$

$\mathcal{B}-a$ is thus hypodissipative on $L^2(\widetilde{\mu}^{-1/2})$ because the norms $\|\cdot\|_{L^2(\widetilde{\mu}^{-1/2})}$ and $\|\cdot\|_{L^2(\mu^{-1/2})}$ are equivalent.

We can now conclude. This splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ fulfills conditions (i), (ii) and (iii) of Lemma 1.5.19. Indeed, it is immediate to check assumption (ii) because $\mathcal{A}$ is a truncation operator. □

1.5.2 Regularization properties of $\left(\mathcal{A}\, e^{\mathcal{B}t}\right)^{(*n)}$

We are now going to show that there exists $n \in \mathbb{N}$ such that $\left(\mathcal{A}\, e^{\mathcal{B}t}\right)^{(*n)}$ has a regularizing effect. In order to get such a result, we are going to use the negative term in the computations done to get the dissipativity of $\mathcal{B}$. Let us state a result which is going to be useful to get an estimate on this negative term.

Lemma 1.5.20 (Fractional Nash inequality)**.** *Consider $\alpha \in (0,2)$. There exists a constant $C > 0$ such that for any $g \in L^1(\mathbb{R}^d) \cap H^{\alpha/2}(\mathbb{R}^d)$, we have:*

$$\int_{\mathbb{R}^d} |g(x)|^2 \, dx \le C \left(\int_{\mathbb{R}^d} |g(x)|\, dx\right)^{\frac{2\alpha}{d+\alpha}} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|g(x)-g(y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx\right)^{\frac{d}{d+\alpha}}.$$

Proof. We use the Plancherel formula to get the following equality for any $R > 0$:

$$\int_{\mathbb{R}^d} |g(x)|^2\, dx = C\left(\int_{|\xi|\le R} |\widehat{g}(\xi)|^2\, d\xi + \int_{|\xi|\ge R} |\widehat{g}(\xi)|^2\, d\xi\right).$$

The first part of the integral can be bounded as follows:

$$\int_{|\xi|\le R} |\widehat{g}(\xi)|^2\, d\xi \le R^d\, \|\widehat{g}\|^2_{L^\infty} \le R^d\, \|g\|^2_{L^1}.$$

As far as the second part is concerned, we use Lemma 1.2.9

$$\int_{|\xi|\ge R} |\widehat{g}(\xi)|^2\, d\xi \le \frac{1}{R^\alpha}\int_{\mathbb{R}^d} |\xi|^\alpha\, |\widehat{g}\,(\xi)|^2\, d\xi = \frac{C}{R^\alpha}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(g(x)-g(y))^2}{|x-y|^{d+\alpha}}\, dy\, dx.$$

We denote

$$a = \|g\|^2_{L^1} \ \text{ et } b = C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(g(x)-g(y))^2}{|x-y|^{d+\alpha}}.$$

and the aim is to minimize the function $\phi(R) := aR^d + bR^{-\alpha}$ to get an optimal inequality. We compute

$$\phi'(R) = 0 \Longleftrightarrow adR^{d-1} - \alpha b\frac{1}{R^{\alpha+1}} = 0 \Longleftrightarrow R = \left(\frac{\alpha b}{ad}\right)^{\frac{1}{d+\alpha}}$$

and

$$\phi\left(\left(\frac{\alpha b}{ad}\right)^{\frac{1}{d+\alpha}}\right) = C a^{\frac{\alpha}{d+\alpha}} b^{\frac{d}{d+\alpha}},$$

which concludes the proof. □

Let us now prove the following lemma which is the cornerstone of the proof of the regularizing effect of $\left(\mathcal{A}\, e^{\mathcal{B}t}\right)^{(*n)}$. We introduce the following measure:

$$m_0(x) := \langle x\rangle^{k_0} \quad \text{with} \quad k_0 < \min(k, \alpha/2).$$

Let us notice that this assumption on k_0 allows us to define $I[m_0^p]$ for any $p \in [1,2]$ and that m_0 satisfies $L^2(\widetilde{\mu}^{-1/2}) \subset L^q(m_0)$ for any $q \in [1,2]$.

Lemma 1.5.21. *There are $b, C > 0$ such that for any p and q, $1 \le p \le q \le 2$, we have:*

$$\forall\, t \geq 0, \quad \left\| e^{\mathcal{B}t}\, f \right\|_{L^q(m_0)} \le \frac{C e^{bt}}{t^{\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)}} \left\| f \right\|_{L^p(m_0)}.$$

Proof. For $p \in [1,2]$, we denote

$$\psi_{m_0,p} := \frac{I[m_0^p]}{p\, m_0^p} + d\,\frac{p-1}{p} - \frac{x\cdot\nabla(m_0^p)}{p\, m_0^p}$$

and we introduce $b \in \mathbb{R}$ such that $\sup_{q\in[1,2]} \psi_{m_0,q} \le b$.

Let us prove that for any $p \in [1,2]$, we have:

$$\forall\, t \geq 0, \quad \|e^{\mathcal{B}t} f\|_{L^p(m_0)} \le e^{bt} \|f\|_{L^p(m_0)}. \tag{1.15}$$

We now do same kind of computations as in the proof of Lemma 1.5.19:

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}f\, |f|^{p-1}\, \mathrm{sign} f\, m_0^p &= \int_{\mathbb{R}^d} I[f]\, |f|^{p-1}\, \mathrm{sign} f\, m_0^p + \int_{\mathbb{R}^d} \mathrm{div}(xf)\, |f|^{p-1}\, \mathrm{sign} f\, m_0^p \\ &=: \widetilde{T}_1 + \widetilde{T}_2.\end{aligned}$$

As far as $\widetilde{T}_1$ is concerned, we introduce the function $\Phi(x) := |x|^p/p$ on $\mathbb{R}^d$ which is convex and its derivative is $\Phi'(x) = |x|^{p-1}\,\mathrm{sign}(x)$. Let us do the following computation:

$$\begin{aligned}&\int_{\mathbb{R}^d} (f(y) - f(x))\, K(x-y)\, dy\, |f|^{p-1}(x)\, \mathrm{sign}(f(x)) \\ &= \int_{\mathbb{R}^d} \left((f(y)-f(x))\,\Phi'(f(x)) + \Phi(f(x)) - \Phi(f(y))\right)\, K(x-y)\, dy \\ &\quad + \int_{\mathbb{R}^d} \left(\Phi(f(y)) - \Phi(f(x))\right)\, K(x-y)\, dy \\ &\le \int_{\mathbb{R}^d} \frac{1}{p}\left(|f|^p(y) - |f|^p(x)\right)\, K(x-y)\, dy = \frac{1}{p} I[|f|^p](x),\end{aligned}$$

where the last inequality comes from the convexity of Φ. We hence deduce that

$$\widetilde{T}_1 \le \frac{1}{p}\int_{\mathbb{R}^d} I[|f|^p]\, m_0^p = \frac{1}{p}\int_{\mathbb{R}^d} |f|^p\, I[m_0^p] = \int_{\mathbb{R}^d} |f|^p\, m_0^p\, \frac{I[m_0^p]}{p\, m_0^p}.$$

Concerning $\widetilde{T}_2$, using an integration by part, we obtain:

$$\widetilde{T}_2 = \int_{\mathbb{R}^d} \left[d\,\frac{p-1}{p} - \frac{x\cdot\nabla(m_0^p)}{p\, m_0^p}\right] |f|^p\, m_0^p.$$

Finally, the previous estimates imply that

$$\int_{\mathbb{R}^d} \mathcal{B}f\, |f|^{p-1}\,\mathrm{sign} f\, m_0^p \le \int_{\mathbb{R}^d} (\psi_{m_0,p} - M\chi_R)\, |f|^p\, m_0^p \le b\int_{\mathbb{R}^d} |f|^p\, m_0^p$$

using the definition of b. This implies the estimate (1.15).

In order to establish the gain of integrability estimate, we have to use the nonpositive term in a sharper way, i.e. not merely the fact that it is nonpositive. It is enough to do that in the simplest case when $p = 2$.

Let us consider a solution f_t of the equation

$$\partial_t f_t = \mathcal{B} f_t\,, \quad f_0 = f \in L^2(m_0).$$

The previous computation involving the function $\Phi(x)$ is simpler in the case $p = 2$ and becomes:

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{B}f\, f\, m_0^2 &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+\alpha}}\, dy\, m_0^2(x)\, dx + \int f^2\, m_0^2\, (\psi_{m_0,2} - M\chi_R)\\
&\le -\frac{1}{2}\int_{\mathbb{R}^d}\int_{|x-y|\le 1} \frac{|f(x)-f(y)|^2}{|x-y|^{d+\alpha}}\, dy\, m_0^2(x)\, dx + b\int f^2\, m_0^2\end{aligned}$$

Let us deal with the negative part of the last inequality.

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{|x-y|\le 1} \frac{|f_t(x)-f_t(y)|^2}{|x-y|^{d+\alpha}}\, dy\, m_0^2(x)\, dx\\
&= \int_{\mathbb{R}^d}\int_{|x-y|\le 1} \frac{|f_t(x)\, m_0(x) - f_t(y)\, m_0(y) + f_t(y)\,(m_0(y)-m_0(x))|^2}{|x-y|^{d+\alpha}}\, dy\, dx\\
&\ge \frac{1}{2}\int_{\mathbb{R}^d}\int_{|x-y|\le 1} \frac{|f_t(x)\, m_0(x) - f_t(y)\, m_0(y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx\\
&\quad - \int_{\mathbb{R}^d}\int_{|x-y|\le 1} \frac{|m_0(x)-m_0(y)|^2}{|x-y|^{d+\alpha}}\, dx\, f_t^2(y)\, dy\\
&\ge \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f_t(x)\, m_0(x) - f_t(y)\, m_0(y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^d}\int_{|x-y|\ge 1} \frac{|f_t(x)\, m_0(x) - f_t(y)\, m_0(y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx\\
&\quad - \int_{\mathbb{R}^d}\int_{|x-y|\le 1} \frac{|m_0(x)-m_0(y)|^2}{|x-y|^{d+\alpha}}\, dx\, f_t^2(y)\, dy\end{aligned}$$

We treat the first term using Lemma 1.5.20 with $g = f_t\, m_0$:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f_t(x)\, m_0(x) - f_t(y)\, m_0(y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx \geq C \left(\int_{\mathbb{R}^d} |f_t|^2\, m_0^2\right)^{\frac{d+\alpha}{d}} \left(\int_{\mathbb{R}^d} |f_t|\, m_0\right)^{-\frac{2\alpha}{d}}. \tag{1.16}$$

We crudely bound the second term from above:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|x-y|\geq 1} \frac{|f_t(x)\, m_0(x) - f_t(y)\, m_0(y)|^2}{|x-y|^{d+\alpha}}\, dy\, m_0^2(x)\, dx \\
&\leq C \left(\int_{\mathbb{R}^d}\int_{|x-y|\geq 1} \frac{|f_t(x)\, m_0(x)|^2}{|x-y|^{d+\alpha}}\, dy\, dx + \int_{\mathbb{R}^d}\int_{|x-y|\geq 1} \frac{|f_t(y)\, m_0(y)|^2}{|x-y|^{d+\alpha}}\, dx\, dy\right) \\
&\leq C \int_{\mathbb{R}^d} |f_t|^2\, m_0^2.
\end{aligned} \tag{1.17}$$

Finally, the third term is bounded using the fact that $\sup_{\bar{B}(y,1)} |\nabla m_0|^2 \leq C\, m_0^2(y)$:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|x-y|\leq 1} \frac{|m_0(x) - m_0(y)|^2}{|x-y|^{d+\alpha}}\, dx\, f_t^2(y)\, dy \\
&\leq C \int_{\mathbb{R}^d}\int_{|x-y|\leq 1} \frac{|x-y|^2 \sup_{\bar{B}(y,1)} |\nabla m_0|^2}{|x-y|^{d+\alpha}}\, dx\, f_t^2(y)\, dy \\
&\leq C \int_{\mathbb{R}^d}\int_{|z|\leq 1} \frac{1}{|z|^{d+\alpha-2}}\, dz\, f_t^2(y)\, m_0^2(y)\, dy \\
&\leq C \int_{\mathbb{R}^d} f_t^2\, m_0^2.
\end{aligned} \tag{1.18}$$

Gathering (1.16), (1.17) et (1.18), we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|x-y|\leq 1} \frac{|f_t(x) - f_t(y)|^2}{|x-y|^{d+\alpha}}\, dy\, m_0^2(x)\, dx \\
&\geq C \left(\int_{\mathbb{R}^d} |f_t|^2\, m_0^2\right)^{\frac{d+\alpha}{d}} \left(\int_{\mathbb{R}^d} |f_t|\, m_0\right)^{-\frac{2\alpha}{d}} - C' \left(\int_{\mathbb{R}^d} f_t^2 m_0^2\right),
\end{aligned} \tag{1.19}$$

for some constants C, $C' > 0$. We introduce the following notations:

$$X(t) := \|f_t\|^2_{L^2(m_0)} \quad \text{and} \quad Y(t) := \|f_t\|_{L^1(m_0)}.$$

On the one hand, if $X_0 \leq (2C'/C)^{d/\alpha}\, Y_0^2$, because of estimate (1.15), we have: $\forall\, t \geq 0$, $X(t)^{1/2} \leq Ce^{bt} X_0^{1/2}$. We hence obtain

$$\forall\, t \geq 0, \quad X(t)^{1/2} \leq C\, e^{bt}\, Y_0.$$

On the other hand, we treat the case $X_0 > (2C'/C)^{d/\alpha}\, Y_0^2$. By the previous step (1.19), we end up with the differential inequality

$$\frac{d}{dt} X(t) \leq -C\, Y(t)^{-\frac{2\alpha}{d}}\, X(t)^{1+\frac{\alpha}{d}} + C'\, X(t). \tag{1.20}$$

We also have from estimate (1.15): $Y(t) \le C\,e^{bt}\,Y(0)$ for any $t \ge 0$. So, we obtain for any $t \in [0,1]$, $Y(t) \le C\,Y(0)$ changing the value of C. Putting this together with (1.20), we obtain:

$$\forall\, t \in [0,1], \quad \frac{d}{dt}X(t) \le -C\,Y_0^{-\frac{2\alpha}{d}}\,X(t)^{1+\frac{\alpha}{d}} + C'\,X(t). \tag{1.21}$$

Let us introduce $\tau := \sup\left\{t \in [0,1] \,:\, X(s) \ge (2C'/C)^{d/\alpha}\,Y_0^2,\ \forall\, s \in [0,t]\right\}$. For any $t \in]0,\tau[$, we have $-1/2\,C\,X(t)^{1+\alpha/d}\,Y_0^{-2\alpha/d} \le -C'X(t)$. Then, using (1.21), we obtain:

$$\forall\, t \in\ (0,\tau), \quad \frac{d}{dt}X(t) \le -\frac{1}{2}\,C\,Y_0^{-\frac{2\alpha}{d}}\,X(t)^{1+\frac{\alpha}{d}},$$

which finally implies

$$\forall\, t \in (0,\tau), \quad X(t) \le \left(\frac{\alpha}{d}\,\frac{C}{2}\,Y_0^{-\frac{2\alpha}{d}}\,t\right)^{-\frac{d}{\alpha}}. \tag{1.22}$$

Moreover, because of estimate (1.15), we get:

$$\forall\, t \in\ [\tau,+\infty), \quad X(t)^{1/2} \le C\,e^{b(t-\tau)}\,X(\tau)^{1/2} \le C\,e^{b(t-\tau)}\,\left(\frac{2C'}{C}\right)^{\frac{d}{2\alpha}}\,Y_0. \tag{1.23}$$

Therefore, gathering inequalities (1.22) an (1.23), we obtain:

$$\forall\, t > 0, \quad X(t)^{\frac{1}{2}} \le C\,t^{-\frac{d}{2\alpha}}\,e^{bt}\,Y_0.$$

As a conclusion, we have

$$\forall\, t > 0, \quad \left\|e^{\mathcal{B}t}\,f\right\|_{L^2(m_0)} \le C\,e^{bt}\,t^{-\frac{d}{2\alpha}}\,\|f\|_{L^1(m_0)}.$$

which means that the operator $e^{\mathcal{B}t}$ is continuous from $L^1(m_0)$ into $L^2(m_0)$.

Let us now consider p and q, $1 \le p \le q \le 2$, $e^{\mathcal{B}t}$ is continuous from $L^p(m_0)$ into $L^q(m_0)$ using Riesz-Thorin interpolation theorem. Moreover, if we denote C_{ab} the norm of $e^{\mathcal{B}t} : L^a(m_0) \to L^b(m_0)$, we get the following estimate:

$$C_{pq} \le C_{22}^{2-2/p}\,C_{11}^{2/q-1}\,C_{12}^{2/p-2/q}$$

and

$$\begin{aligned} C_{22}^{2-2/p}\,C_{11}^{2/q-1}\,C_{12}^{2/p-2/q} &= C\,e^{bt(2-2/p)}\,e^{bt(2/q-1)}\,e^{bt(2/p-2/q)}\,t^{-d/(2\alpha)\,(2/p-2/q)} \\ &= \frac{C\,e^{bt}}{t^{\frac{d}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)}}, \end{aligned}$$

which yields the result. □

Using the same method as in [51], we can deduce the following corollary:

Corollary 1.5.22. *There exists a constant C such that for any p and q, $1 \le p \le q \le 2$, we have:*

$$\forall\, t \ge 0, \quad \|T_{\ell_0}(t) f\|_{L^q(m_0)} \le C\, \frac{t^{\ell_0 - 1}\, e^{bt}}{t^{\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)}}\, \|f\|_{L^p(m_0)}$$

where $\ell_0 = E\left[\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)\right] + 1$.

1.5.3 Proof of the main result

As a conclusion, we can now apply Theorem 1.3.10 with $E = L^2(\widetilde{\mu}^{-1/2})$ and $\mathcal{E} = L^1(m)$. Hypothesis (1) comes from Corollary 1.4.18. Hypothesis (2)-(i) and (2)-(ii) come from Lemma 1.5.19. We can also prove that assumption (2)-(iii) is satisfied.

Indeed, we can check by an immediate computation that we have the following estimate for any function f: $\| \mathcal{A} f\|_{L^q(m)} \le C\, \|\mathcal{A} f\|_{L^q(m_0)}$. Moreover, we have that $L^p(m) \subset L^p(m_0)$ with continuous embedding (because $k_0 < k$). Using these two last facts and Corollary 1.5.22, we can deduce that for any p and q, $1 \le p \le q \le 2$, we have:

$$\forall\, t \ge 0, \quad \|T_{\ell_0}(t) f\|_{L^q(m)} \le C\, \frac{t^{\ell_0 - 1}\, e^{bt}}{t^{\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)}}\, \|f\|_{L^p(m)}. \tag{1.24}$$

Moreover, we can show that $\| \mathcal{A} f\|_{L^2(\widetilde{\mu}^{-1/2})} \le C\, \|\mathcal{A} f\|_{L^2(m)}$. Finally, using this last estimate combined with (1.24) with $p = 1$, $q = 2$ and denoting $\gamma := \frac{d}{2\alpha} - E\left(\frac{d}{2\alpha}\right)$, we obtain:

$$\|T_{\ell_0}(t)\|_{L^1(m) \to L^2(\widetilde{\mu}^{-1/2})} \le C\, \frac{e^{bt}}{t^\gamma},$$

with $\gamma \in [0, 1)$, which implies that (2)-(iii) is fulfilled.

We can conclude that Theorem 1.1.6 holds.

Remark 1.5.23. *To obtain a similar result as Theorem 1.1.6 in $L^p(\langle x \rangle^k)$ with $p \in (1, 2]$, we need a very restrictive assumption: $d\,(1 - 1/p) < k < \alpha$. Indeed, it implies that the limit at infinity of $\psi_{m,p}$ is negative, which allows us to get the dissipativity of $\mathcal{B} - a$ in $L^p(\langle x \rangle^k)$ for any $a > d\,(1 - 1/p) - k$. The rest of the proof can be done in the same way.*

Chapitre 2

Exponential convergence to equilibrium for the homogeneous Boltzmann equation for hard potentials without cut-off

Résumé. Dans cette partie, nous abordons le problème du comportement en temps long des solutions de l'équation de Boltzmann homogène en espace. Les interactions considérées sont les potentiels durs (non tronqués et non lissés), nous ne traitons donc que des singularités angulaires modérées. Nous prouvons une convergence vers l'équilibre exponentielle en temps, améliorant ainsi les résultats de Villani obtenus dans [94] où un retour polynomial vers l'équilibre est prouvé. La base de la preuve est l'étude de l'équation linéarisée pour laquelle nous prouvons une nouvelle estimation de trou spectral dans un espace L^1 à poids polynomial, tirant parti de la théorie développée par Gualdani et al. dans [51]. Nous obtenons notre résultat final en combinant cette nouvelle estimation spectrale avec de nouvelles estimations bilinéaires sur l'opérateur de collision.

Abstract. This part deals with the long time behavior of solutions to the spatially homogeneous Boltzmann equation. The interactions considered are the so-called (non cut-off and non mollified) hard potentials, we thus only deal with a moderate angular singularity. We prove an exponential in time convergence towards the equilibrium, improving results of Villani from [94] where a polynomial decay to equilibrium is proven. The basis of the proof is the study of the linearized equation for which we prove a new spectral gap estimate in a L^1 space with a polynomial weight by taking advantage of the theory of enlargement of the functional space for the semigroup decay developed by Gualdani et al. in [51]. We then get our final result by combining this new spectral gap estimate with bilinear estimates on the collisional operator that we establish.

2.1 Introduction

2.1.1 The model

In the present paper, we investigate the asymptotic behavior of solutions to the spatially homogeneous Boltzmann equation without angular cut-off, that is, for long-range interactions. Previous works have shown that these solutions converge towards the Maxwellian equilibrium with a polynomial rate when time goes to infinity. Here, we are interested in improving the rate of convergence and we show an exponential decay to equilibrium.

We consider particles described by their space homogeneous distribution density $f = f(t,v)$. We hence study the so-called spatially homogeneous Boltzmann equation:

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad v \in \mathbb{R}^3, \quad t \geq 0. \tag{2.1}$$

The Boltzmann collision operator is defined as

$$Q(g,f) := \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,[g'_*f' - g_*f]\;d\sigma\,dv_*.$$

Here and below, we are using the shorthand notations $f = f(v)$, $g_* = g(v_*)$, $f' = f(v')$ and $g'_* = g(v'_*)$. In this expression, v, v_* and v', v'_* are the velocities of a pair of particles before and after collision. We make a choice of parametrization of the set of solutions to the conservation of momentum and energy (physical law of elastic collisions):

$$v + v_* = v' + v'_*,$$

$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

so that the post-collisional velocities are given by:

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma, \quad \sigma \in \mathbb{S}^2.$$

The Boltzmann collision kernel $B(v-v_*,\sigma)$ only depends on the relative velocity $|v-v_*|$ and on the deviation angle θ through $\cos\theta = \langle \kappa,\sigma\rangle$ where $\kappa = (v-v_*)/|v-v_*|$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^3$. By a symmetry argument, one can always reduce to the case where $B(v-v_*,\sigma)$ is supported on $\langle\kappa,\sigma\rangle \geq 0$ i.e $0 \leq \theta \leq \pi/2$. So, without loss of generality, we make this assumption.

In this paper, we shall be concerned with the case when the kernel B satisfies the following conditions:

- it takes product form in its arguments as

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)\,b(\cos\theta); \tag{2.2}$$

- the angular function b is locally smooth, and has a nonintegrable singularity for $\theta \to 0$, it satisfies for some $c_b > 0$ and $s \in (0,1/2)$ (moderate angular singularity)

$$\forall\,\theta \in (0,\pi/2], \quad \frac{c_b}{\theta^{1+2s}} \leq \sin\theta\, b(\cos\theta) \leq \frac{1}{c_b\,\theta^{1+2s}}; \tag{2.3}$$

– the kinetic factor Φ satisfies

$$\Phi(|v-v_*|) = |v-v_*|^\gamma \quad \text{with} \quad \gamma \in (0,1), \tag{2.4}$$

this assumption could be relaxed to assuming only that Φ satisfies $\Phi(\cdot) = C_\Phi\, |\cdot|^\gamma$ for some $\mathbb{C}_\Phi > 0$.

Our main physical motivation comes from particles interacting according to a repulsive potential of the form

$$\phi(r) = r^{-(p-1)}, \quad p \in (2, +\infty). \tag{2.5}$$

The assumptions made on B throughout the paper include the case of potentials of the form (2.5) with $p > 5$. Indeed, for repulsive potentials of the form (2.5), the collision kernel cannot be computed explicitly but Maxwell [62] has shown that the collision kernel can be computed in terms of the interaction potential ϕ. More precisely, it satisfies the previous conditions (2.2), (2.3) and (2.4) in dimension 3 (see [29, 31, 93]) with $s := \frac{1}{p-1} \in (0,1)$ and $\gamma := \frac{p-5}{p-1} \in (-3,1)$.

One traditionally calls *hard potentials* the case $p > 5$ (for which $0 < \gamma < 1$), *Maxwell molecules* the case $p = 5$ (for which $\gamma = 0$) and *soft potentials* the case $2 < p < 5$ (for which $-3 < \gamma < 0$). We can hence deduce that our assumptions made on B include the case of hard potentials.

The equation (2.1) preserves mass, momentum and energy. Indeed, at least formally, we have:

$$\int_{\mathbb{R}^3} Q(f,f)(v)\, \varphi(v)\, dv = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2;$$

from which we deduce that a solution f_t to the equation (2.1) is conservative, meaning that

$$\forall\, t \geq 0, \quad \int_{\mathbb{R}^3} f(t,v)\, \varphi(v)\, dv = \int_{\mathbb{R}^3} f_0(v)\, \varphi(v)\, dv \quad \text{for} \quad \varphi(v) = 1, v, |v|^2. \tag{2.6}$$

We introduce the entropy $H(f) = \int_{\mathbb{R}^3} f \,\log(f)$ and the entropy production $D(f)$. Boltzmann's H theorem asserts that

$$\frac{d}{dt} H(f) = -D(f) \leq 0 \tag{2.7}$$

and states that any equilibrium (i.e any distribution which maximizes the entropy) is a Maxwellian distribution $\mu_{\rho,u,T}$ for some $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$:

$$\mu^f(v) = \mu_{\rho,u,T}(v) = \frac{\rho\, e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{3/2}},$$

where ρ, u and T are the mass, momentum and temperature of the gas:

$$\rho := \rho_f = \int_{\mathbb{R}^3} f(v)\, dv, \quad u := u_f = \frac{1}{\rho} \int_{\mathbb{R}^3} f(v)\, v\, dv, \quad T := T_f = \frac{1}{3\rho} \int_{\mathbb{R}^3} f(v)\, |v-u|^2\, dv.$$

Thanks to the conservation properties of the equation (2.6), the following equalities hold:

$$\rho_f = \rho_{f_0}, \quad u_f = u_{f_0}, \quad T_f = T_{f_0}$$

where f_0 is the initial datum.

Moreover, a solution f_t of the Boltzmann equation is expected to converge towards the Maxwellian distribution $\mu_{\rho,u,T}$ when $t \to +\infty$.

In this paper, we only consider the case of an initial datum satisfying

$$\rho_{f_0} = 1, \quad u_{f_0} = 0, \quad T_{f_0} = 1, \tag{2.8}$$

one can always reduce to this situation (see [94]). We then denote μ the Maxwellian with same mass, momentum and energy of f_0: $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$.

2.1.2 Function spaces and notations

We introduce some notations about weighted L^p spaces. For some given Borel weight function $m \geq 0$ on $\mathbb{R}^3$, we define the Lebesgue weighted space $L^p(m)$, $1 \leq p \leq +\infty$, as the Lebesgue space associated to the norm

$$\|h\|_{L^p(m)} := \|h\, m\|_{L^p}.$$

We also define the weighted Sobolev space $W^{s,p}(m)$, $s \in \mathbb{N}$, $1 \leq p < +\infty$, as the Sobolev space associated to the norm

$$\|h\|_{W^{s,p}(m)} := \left(\sum_{|\alpha| \ \leq s} \| \ \partial^\alpha h\|^p_{L^p(m)} \right)^{1/p}.$$

Throughout this paper, we will use the same notation C for positive constants that may change from line to line. Moreover, the notation $A \approx B$ will mean that there exist two constants C_1, $C_2 > 0$ such that $C_1\, A \leq B \leq C_2\, A$.

2.1.3 Main results and known results

Convergence to equilibrium

We first state our main result on exponential convergence to equilibrium.

Theorem 2.1.24. *Consider a collision kernel B satisfying conditions (2.2), (2.3), (2.4) and f_0 a nonnegative distribution with finite mass, energy and entropy:*

$$f_0 \geq 0, \quad \int_{\mathbb{R}^3} f_0(v)\,(1+|v|^2)\,dv < \infty, \quad \int_{\mathbb{R}^3} f_0(v)\, \log(f_0(v))\,dv < \infty$$

and satisfying (2.8). Then, if f_t is a smooth solution (see Definition 2.1.25) to the equation (2.1) with initial datum f_0, there exists a constant $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_{L^1} \leq C\, e^{-\lambda\, t}$$

where $\lambda > 0$ is defined in Theorem 2.1.27.

We improve a polynomial result of Villani [94] and generalize to our context similar exponential results known for simplified models. Mouhot in [75] proved such a result for the spatially homogeneous Boltzmann equation with hard potentials and Grad's cut-off. Carrapatoso in [25] recently proved exponential decay to equilibrium for the homogeneous Landau equation with hard potentials which is the grazing collisions limit of the model we study in the present paper. Let us also mention the paper of Gualdani et al. [51] where an exponential decay to equilibrium is proved for the inhomogeneous Boltzmann equation for hard spheres (see also [66, 70, 67] for related works).

It is a known fact that our equation (2.1) admits solutions which are conservative and satisfy some suitable properties of regularity, we will call them *smooth solutions*. We here precise the meaning of this term and give an overview of results on the Cauchy theory of our equation.

Definition 2.1.25. *Let f_0 be a nonnegative function defined on $\mathbb{R}^3$ with finite mass, energy and entropy. We shall say that $(t,v) \mapsto f(t,v)$ is a* smooth solution *to the equation (2.1) if the following conditions are fulfilled:*

– $f \geq 0$, $f \in \mathcal{C}(\mathbb{R}^+, L^1)$;

– *for any* $t \geq 0$,

$$\int_{\mathbb{R}^3} f(t,v)\,\varphi(v)\,dv = \int_{\mathbb{R}^3} f_0(v)\,\varphi(v)\,dv \quad \textit{for} \quad \varphi(v) = 1, v, |v|^2$$

and

$$\int_{\mathbb{R}^3} f(t,v)\,\log(f(t,v))\,dv + \int_0^t D(f(s,\cdot))\,ds \leq \int_{\mathbb{R}^3} f_0(v)\,\log(f_0(v))\,dv\,;$$

where $D(f)$ is the entropy production defined in (2.7);

– *for any $\varphi \in \mathcal{C}^1(\mathbb{R}^+, \mathscr{D}(\mathbb{R}^3))$ and for any $t \geq 0$,*

$$\int_{\mathbb{R}^3} f(t,v)\,\varphi(t,v)\,dv = \int_{\mathbb{R}^3} f_0(v)\,\varphi(0,v)\,dv + \int_0^t \int_{\mathbb{R}^3} f(\tau,v)\,\partial_t\varphi(\tau,v)\,dv\,d\tau + \int_0^t \int_{\mathbb{R}^3} Q(f,f)(\tau,v)\,\varphi(\tau,v)\,dv\,d\tau$$

where the last integral is define through the following formula

$$\int_{\mathbb{R}^3} Q(f,f)(v)\,\varphi(v)\,dv = \frac{1}{4} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,[f'_*f'-f_*f]\,(\varphi+\varphi_*-\varphi'-\varphi'_*)\,d\sigma\,dv_*\,dv;$$

– *for any $t_0 > 0$ and for any $\ell \in \mathbb{R}^+$,*

$$\sup_{t\geq t_0} \|f(t,\cdot)\|_{L^1(\langle v\rangle^\ell)} < \infty\,; \tag{2.9}$$

– *for any $t_0 > 0$ and for any N, $\ell \in \mathbb{R}^+$,*

$$\sup_{t\geq t_0} \|f(t,\cdot)\|_{H^N(\langle v\rangle^\ell)} < \infty. \tag{2.10}$$

Such a solution is known to exist. The problem of existence of solutions was first studied by Arkeryd in [10] where existence of solutions is proven for not too soft potentials, that is $\gamma > -1$ (Goudon [46] and Villani [92] then improved this result enlarging the class of γ considered). We mention that uniqueness of solution for hard potentials can be proved under some more restrictive conditions on the initial datum, see the paper of Desvillettes and Mouhot [37] where f_0 is supposed to be regular ($f_0 \in W^{1,1}(\langle v\rangle^2)$) and the paper of Fournier and Mouhot [43] where f_0 is supposed to be localized ($\int_{\mathbb{R}^3} f_0 \, e^{a|v|^\gamma} dv < \infty$ for some $a > 0$) for hard potentials.

Concerning the moment production property, it was discovered by Elmroth [41] and Desvillettes [36] and improved by Wennberg [100], which justifies our point (2.9) in the definition of a smooth solution. We here point out the fact that this property is not anymore true for Maxwell molecules or soft potentials. As a consequence, our method, which relies partially on this property, works only for hard potentials.

Finally, we mention papers where regularization results are proven for "true" (that is non mollified) physical potentials: [1] by Alexandre et al. and [33] by Chen and He where the initial datum is supposed to have finite energy and entropy, [14] by Bally and Fournier where only the $2D$ case is treated and [42] by Fournier under others conditions on the initial datum. Theorem 1.4 from [33] explains our point (2.10).

We now recall previous results on convergence to equilibrium for solutions to equation (2.1). It was first studied by Carlen and Carvalho [23, 24] and then by Toscani and Villani [87]. Up to now, the best rate of convergence in our case was obtained by Villani in [94]:

Theorem 2.1.26. *Let us consider f_t a smooth solution to (2.1) with an initial datum f_0 satisfying (2.8) with finite entropy. Then f_t satisfies the following polynomial decay to equilibrium: for any $t_0 > 0$ and any $\varepsilon > 0$, there exists $C_{t_0,\varepsilon} > 0$ such that*

$$\forall\, t \geq t_0, \quad \|f_t - \mu\|_{L^1} \leq C_{t_0,\varepsilon} \, t^{-\frac{1}{\varepsilon}}.$$

This result comes from [94, Theorem 4.1] which states that if f is a function which satisfies the following lowerbound

$$\forall\, v \in \mathbb{R}^3, \quad f(v) \geq K_0 \, e^{-A_0 |v|^{q_0}} \quad \text{with} \quad K_0, A_0 > 0, \, q_0 \geq 2 \tag{2.11}$$

then for any $\varepsilon > 0$, there exists an explicit constant $K_\varepsilon > 0$ such that

$$D(f) \geq K_\varepsilon \, H(f|\mu)^{1+\varepsilon}. \tag{2.12}$$

It is a result from Mouhot [73, Theorem 1.2] that the lowerbound (2.11) holds for any smooth solution f_t of our equation (2.1). Let us mention that lowerbounds of solutions were first studied by Carleman [21] (for hard spheres) and then by Pulvirenti and Wennberg [80] (for hard potentials with cut-off). Finally, Mouhot [73] extended these results to the spatially inhomogeneous case without cut-off. We here state Theorem 1.2 from [73] that we use: for any $t_0 > 0$ and for any exponent q_0 such that

$$q_0 > 2 \frac{\log\left(2 + \frac{2s}{1-s}\right)}{\log 2},$$

a smooth solution f_t to (2.1) satisfies

$$\forall\, t \geq t_0, \quad \forall\, v \in \mathbb{R}^3, \quad f(t,v) \geq K_0\, e^{-A_0 |v|^{q_0}}$$

for some K_0, $A_0 > 0$.

We can then deduce that the conclusion of Theorem 2.1.26 holds using the Csiszár-Kullback-Pinsker inequality $\|f-\mu\|_{L^1} \leq \sqrt{2H(f|\mu)}$ combined with the result of Villani (2.12).

Let us here emphasize that the method of Villani to prove the polynomial convergence towards equilibrium is purely nonlinear. Ours is based on the study of the linearized equation.

The linearized equation

We introduce the linearized operator. Considering the linearization $f = \mu + h$, we obtain at first order the linearized equation around the equilibrium μ

$$\partial_t h = \mathcal{L} h := Q(\mu, h) + Q(h, \mu), \tag{2.13}$$

for $h = h(t,v)$, $v \in \mathbb{R}^3$. The null space of the operator $\mathcal{L}$ is the 5-dimensional space

$$\mathcal{N}(\mathcal{L}) = \mathrm{Span}\left\{\mu, \mu\, v_1, \mu\, v_2, \mu\, v_3, \mu\, |v|^2\right\}. \tag{2.14}$$

Our strategy is to combine the polynomial convergence to equilibrium and a spectral gap estimate on the linearized operator to show that if the solution enters some stability neighborhood of the equilibrium, then the convergence is exponential in time. Previous results on spectral gap estimates hold only in $L^2(\mu^{-1/2})$ and the Cauchy theory for the nonlinear Boltzmann equation is constructed in L^1-spaces with polynomial weight. In order to link the linear and the nonlinear theories, our approach consists in proving new spectral gap estimates for the linearized operator $\mathcal{L}$ in spaces of type $L^1(\langle v\rangle^k)$. To do that, we exhibit a convenient splitting of the linearized operator in such a way that we may use the abstract theorem from [51] which allows us to enlarge the space of spectral estimates of a given operator.

Here is the result we obtain on the linearized equation which provides a constructive spectral gap estimate for $\mathcal{L}$ in $L^1(\langle v\rangle^k)$ and which is the cornerstone of the proof of Theorem 2.1.24.

Theorem 2.1.27. *Let $k > 2$ and a collision kernel B satisfying (2.2), (2.3) and (2.4). Consider the linearized Boltzmann operator $\mathcal{L}$ defined in (2.13). Then, for any positive $\lambda < \min(\lambda_0, \lambda_k)$ (where λ_0 is the spectral gap of $\mathcal{L}$ in $L^2(\mu^{-1/2})$ defined in Proposition 2.2.28 and λ_k is a constant depending on k defined in Lemma 2.2.34), there exists an explicit constant $C_\lambda > 0$, such that for any $h \in L^1(\langle v\rangle^k)$, we have the following estimate*

$$\forall\, t \geq 0, \quad \|S_{\mathcal{L}}(t)h - \Pi h\|_{L^1(\langle v\rangle^k)} \leq C_\lambda\, e^{-\lambda t} \|h - \Pi h\|_{L^1(\langle v\rangle^k)}, \tag{2.15}$$

where $S_{\mathcal{L}}(t)$ denotes the semigroup of $\mathcal{L}$ and Π the projection onto $\mathcal{N}(\mathcal{L})$.

Let us briefly review the existing results concerning spectral gap estimates for $\mathcal{L}$. Pao [78] studied spectral properties of the linearized operator for hard potentials by non-constructive and very technical means. This article was reviewed by Klaus [58]. Then, Baranger and Mouhot gave the first explicit estimate on this spectral gap in [15] for hard potentials ($\gamma > 0$). If we denote $\mathcal{D}$ the Dirichlet form associated to $-\mathcal{L}$:

$$\mathcal{D}(h) := \int_{\mathbb{R}^3} (-\mathcal{L}h)\, h\, \mu^{-1},$$

and $\mathcal{N}(\mathcal{L})^\perp$ the orthogonal of $\mathcal{N}(\mathcal{L})$ defined in (2.14) and Π the projection onto $\mathcal{N}(\mathcal{L})$, the Dirichlet form $\mathcal{D}$ satisfies

$$\forall\, h \in \mathcal{N}(\mathcal{L})^\perp, \quad \mathcal{D}(h) \geq \lambda_0 \, \|h\|^2_{L^2(\mu^{-1/2})}, \tag{2.16}$$

for some constructive constant $\lambda_0 > 0$. This result was then improved by Mouhot [74] and later by Mouhot and Strain [77]. In the last paper, it was conjectured that a spectral gap exists if and only if $\gamma + 2s \geq 0$. This conjecture was finally proven by Gressman and Strain in [50].

Another question would be to obtain similar results in other spaces: L^p spaces with $1 < p \leq 2$ and a polynomial weight or L^p spaces with $1 \leq p \leq 2$ and a stretched exponential weight. Our computations do not allow to conclude in those cases, more precisely, we are not able to do the computations which allow to obtain the suitable splitting of the linear operator in order to apply the theorem of enlargement of the space of spectral estimates. As a consequence, we can not prove the existence of a spectral gap on those spaces. However, we believe that such results may hold.

We here point out that the knowledge of a spectral gap estimate in $L^1(\langle v\rangle^k)$ for the fractional Fokker-Planck equation (see [90]) is consistent with our result. Indeed, the behavior of the Boltzmann collision operator has been widely conjectured to be that of a fractional diffusion (see [35, 46, 92]).

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2.2 The linearized equation

Here and below, we denote $m(v) := \langle v\rangle^k$ with $k > 2$. The aim of the present section is to prove Theorem 2.1.27. To do that, we exhibit a splitting of the linearized operator into two parts, one which is bounded and the second one which is dissipative. We can then apply the abstract theorem of enlargement of the functional space of the semigroup decay from Gualdani et al. [51] (see Subsection 2.2.4).

2.2.1 Notations

We now introduce notations about spectral theory of unbounded operators. For a given real number $a \in \mathbb{R}$, we define the half complex plane

$$\Delta_a := \{z \in \mathbb{C},\, \Re e\, z > a\}\,.$$

For some given Banach spaces $(E, \|\cdot\|_E)$ and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, we denote by $\mathscr{B}(E,\mathcal{E})$ the space of bounded linear operators from E to $\mathcal{E}$ and we denote by $\|\cdot\|_{\mathscr{B}(E,\mathcal{E})}$ or $\|\cdot\|_{E\to\mathcal{E}}$ the associated norm operator. We write $\mathscr{B}(E) = \mathscr{B}(E,E)$ when $E = \mathcal{E}$. We denote by $\mathscr{C}(E,\mathcal{E})$ the space of closed unbounded linear operators from E to $\mathcal{E}$ with dense domain, and $\mathscr{C}(E) = \mathscr{C}(E,E)$ in the case $E = \mathcal{E}$.

For a Banach space X and $\Lambda \in \mathscr{C}(X)$ we denote by $S_\Lambda(t)$, $t \geq 0$, its semigroup, by $\mathrm{D}(\Lambda)$ its domain, by $\mathrm{N}(\Lambda)$ its null space and by $\mathrm{R}(\Lambda)$ its range. We also denote by $\Sigma(\Lambda)$ its spectrum, so that for any z belonging to the resolvent set $\rho(\Lambda) := \mathbb{C}\backslash\Sigma(\Lambda)$ the operator $\Lambda - z$ is invertible and the resolvent operator

$$\mathcal{R}_\Lambda(z) := (\Lambda - z)^{-1}$$

is well-defined, belongs to $\mathscr{B}(X)$ and has range equal to $\mathrm{D}(\Lambda)$. An eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be isolated if

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C},\ |z-\xi| \leq r\} = \{\xi\} \ \text{ for some } r > 0.$$

In the case when ξ is an isolated eigenvalue, we may define $\Pi_{\Lambda,\xi} \in \mathscr{B}(X)$ the associated spectral projector by

$$\Pi_{\Lambda,\xi} := -\frac{1}{2i\pi}\int_{|z-\xi|=r'} \mathcal{R}_\Lambda(z)\,dz$$

with $0 < r' < r$. Note that this definition is independent of the value of r' as the application $\mathbb{C}\setminus\Sigma(\Lambda) \to \mathscr{B}(X)$, $z \to \mathcal{R}_\Lambda(z)$ is holomorphic. For any $\xi \in \Sigma(\Lambda)$ isolated, it is well-known (see [57] paragraph III-6.19) that $\Pi^2_{\Lambda,\xi} = \Pi_{\Lambda,\xi}$, so that $\Pi_{\Lambda,\xi}$ is indeed a projector.

When moreover the so-called “algebraic eigenspace" $\mathrm{R}(\Pi_{\Lambda,\xi})$ is finite dimensional we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$.

2.2.2 Spectral gap in $L^2(\mu^{-1/2})$

We here state a direct consequence of inequality (2.16) from [15], which gives us a spectral gap estimate in $L^2(\mu^{-1/2})$.

Proposition 2.2.28. *There is a constructive constant $\lambda_0 > 0$ such that*

$$\forall\, t \geq 0, \quad \forall\, h \in L^2(\mu^{-1/2}), \quad \|S_{\mathcal{L}}(t)h - \Pi h\|_{L^2(\mu^{-1/2})} \leq e^{-\lambda_0 t}\|h - \Pi h\|_{L^2(\mu^{-1/2})}.$$

2.2.3 Splitting of the linearized operator

We first split the linearized operator $\mathcal{L}$ defined in (2.13) into two parts, separating the grazing collisions and the cut-off part, we define

$$b_\delta := \mathbb{1}_{\theta \leq \delta}\, b \quad \text{and} \quad b_\delta^c := \mathbb{1}_{\theta \geq \delta}\, b$$

for some $\delta \in (0,1)$ to be chosen later, it induces the following splitting of $\mathcal{L}$:

$$\begin{aligned}\mathcal{L}h &= \mathcal{L}_\delta h + \mathcal{L}_\delta^c h \\ &=: \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left[\mu'_* h' - \mu_* h + h'_* \mu' - h_* \mu\right]\, b_\delta(\cos\theta)|v - v_*|^\gamma\, d\sigma\, dv_* \\ &\quad + \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left[\mu'_* h' - \mu_* h + h'_* \mu' - h_* \mu\right]\, b_\delta^c(\cos\theta)|v - v_*|^\gamma\, d\sigma\, dv_*.\end{aligned}$$

In the rest of the paper, we shall use the notations

$$B_\delta(v - v*, \sigma) := b_\delta(\cos\theta)\, |v - v_*|^\gamma \quad \text{and} \quad B_\delta^c(v - v_*, \sigma) := b_\delta^c(\cos\theta)\, |v - v_*|^\gamma.$$

As far as the cut-off part is concerned, our strategy is similar as the one adopted in [51] for hard-spheres. For any $\varepsilon \in (0,1)$, we consider $\Theta_\varepsilon \in \mathcal{C}^\infty$ bounded by one, which equals one on

$$\left\{|v| \leq \varepsilon^{-1} \text{ and } 2\varepsilon \leq |v - v_*| \leq \varepsilon^{-1} \text{ and } |\cos\,\theta| \leq 1 - 2\varepsilon\right\}$$

and whose support is included in

$$\left\{|v| \leq 2\varepsilon^{-1} \text{ and } \varepsilon \leq |v - v_*| \leq 2\varepsilon^{-1} \text{ and } |\cos\,\theta| \leq 1 - \varepsilon\right\}.$$

We then denote the truncated operator

$$\mathcal{A}_{\delta,\varepsilon}(h) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} \Theta_\varepsilon \left[\mu'_*\, h' \,+\, \mu'\, h'_* - \mu\, h_*\right]\, b_\delta^c(\cos\theta)|v - v_*|^\gamma\, d\sigma\, dv_*$$

and the corresponding remainder operator

$$\mathcal{B}_{\delta,\varepsilon}^c(h) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} (1 - \Theta_\varepsilon) \left[\mu'_*\, h' \,+\, \mu'\, h'_* - \mu\, h_*\right]\, b_\delta^c(\cos\theta)|v - v_*|^\gamma\, d\sigma\, dv_*.$$

We also introduce

$$\nu_\delta(v) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mu_*\, b_\delta^c(\cos\theta)|v - v_*|^\gamma\, d\sigma\, dv_*,$$

so that we have the following splitting: $\mathcal{L}_\delta^c = \mathcal{A}_{\delta,\varepsilon} + \mathcal{B}_{\delta,\varepsilon}^c - \nu_\delta$.

Moreover, ν_δ satisfies

$$\nu_\delta(v) = K_\delta\, (\mu * |\;\cdot|^\gamma)(v)$$

with

$$K_\delta := \int_{\mathbb{S}^2} b_\delta^c(\cos\theta)\, d\sigma \approx \int_\delta^{\pi/2} b(\cos\theta)\, \sin\theta\, d\theta \approx \delta^{-2s} - \left(\frac{\pi}{2}\right)^{-2s} \xrightarrow[\delta \to 0]{} +\infty$$

using the spherical coordinates to get the second equality and (2.3) to get the final one; and

$$(\mu * |\cdot|^\gamma)(v) \approx \langle v\rangle^\gamma.$$

We finally define

$$\mathcal{B}_{\delta,\varepsilon} := \mathcal{L}_\delta + \mathcal{B}^c_{\delta,\varepsilon} - \nu_\delta$$

so that $\mathcal{L} = \mathcal{A}_{\delta,\varepsilon} + \mathcal{B}_{\delta,\varepsilon}$.

Dissipativity properties

Lemma 2.2.29. *There exists a function $\varphi_k(\delta)$ depending on k and tending to 0 as δ tends to 0 such that for any $h \in L^1(\langle v\rangle^\gamma m)$, the following estimate holds:*

$$\int_{\mathbb{R}^3} \mathcal{L}_\delta(h)\,\mathrm{sign}(h)\,m\,dv \le \varphi_k(\delta)\,\|h\|_{L^1(\langle v\rangle^\gamma m)}. \tag{2.17}$$

Proof. Let us first introduce a notation which is going to be useful in the sequel of the proof:

$$\kappa_\delta := \int_0^{\pi/2} b_\delta(\cos\theta)\,\sin^2(\theta)\,d\theta = \int_0^\delta b(\cos\theta)\,\sin^2(\theta)\,d\theta \approx \delta^{1-2s} \xrightarrow[\delta\to 0]{} 0, \tag{2.18}$$

where the last equality comes from (2.3). We here underline the fact that considering a moderate singularity, meaning $s \in (0,1/2)$, is here needed to get the convergence of κ_δ to 0 as δ goes to 0.

We split $\mathcal{L}_\delta$ into two parts in the following way:

$$\begin{aligned}
\mathcal{L}_\delta h &= \int_{\mathbb{R}^3\times\mathbb{S}^2} \left[\mu'_*\,h' - \mu_*\,h\right]\,b_\delta(\cos\theta)|v-v_*|^\gamma\,d\sigma\,dv_* \\
&\quad + \int_{\mathbb{R}^3\times\mathbb{S}^2} \left[h'_*\,\mu' - h_*\,\mu\right]\,b_\delta(\cos\theta)|v-v_*|^\gamma\,d\sigma\,dv_* \\
&=: \mathcal{L}^1_\delta h + \mathcal{L}^2_\delta h,
\end{aligned}$$

this splitting corresponds to the splitting of $\mathcal{L}_\delta$ as $Q_\delta(\mu,h) + Q_\delta(h,\mu)$ if Q_δ denotes the collisional operator associated to the kernel B_δ.

We first deal with $\mathcal{L}^1_\delta$. Let us recall that we have $\mu\,\mu_* = \mu'\,\mu'_*$. In the following computation, we denote $g := h\,\mu^{-1}$:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{L}^1_\delta(h)\,\mathrm{sign}(h)\,m\,dv \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g'-g\right]\,\mathrm{sign}(g)\,m\,d\sigma\,dv_*\,dv \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g'-g\right]\,\left[\mathrm{sign}(g)-\mathrm{sign}(g')\right]\,m\,d\sigma\,dv_*\,dv \\
&\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g'-g\right]\,\mathrm{sign}(g')\,m\,d\sigma\,dv_*\,dv \\
&\le \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g'-g\right]\,\mathrm{sign}(g')\,m\,d\sigma\,dv_*\,dv,
\end{aligned}$$

where we used that for any $a,\ b \in \mathbb{R}$, $(a-b)(\mathrm{sign}(a) - \mathrm{sign}(b)) \le 0$ to get the last inequality.

Remark 2.2.30. *We here emphasize that this computation is particularly convenient in the L^1 case since sign$(h) =$ sign(g). In the L^p case, it is trickier and for now, we are not able to adapt it to get the wanted estimates.*

We now use the classical pre-post collisional change of variables to pursue the computation:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{L}^1_\delta(h)\,\mathrm{sign}(h)\,m\,dv \le \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g-g'\right]\,\mathrm{sign}(g)\,m'\,d\sigma\,dv_*\,dv \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g-g'\right]\,\mathrm{sign}(g)\,(m'-m)\,d\sigma\,dv_*\,dv \\
&\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g-g'\right]\,\mathrm{sign}(g)\,m\,d\sigma\,dv_*\,dv. \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g-g'\right]\,\mathrm{sign}(g)\,(m'-m)\,d\sigma\,dv_*\,dv \\
&\quad - \int_{\mathbb{R}^3} \mathcal{L}^1_\delta(h)\,\mathrm{sign}(h)\,m\,dv.
\end{aligned}$$

We hence deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{L}^1_\delta(h)\,\mathrm{sign}(h)\,m\,dv \\
&\le \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,\left[g-g'\right]\,\mathrm{sign}(g)\,(m'-m)\,d\sigma\,dv_*\,dv \\
&\le \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu\,\mu_*\,|g|\,|m'-m|\,d\sigma\,dv_*\,dv \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu_*\,|h|\,|m'-m|\,d\sigma\,dv_*\,dv.
\end{aligned}$$

We now estimate the difference $|m'-m|$:

$$|m'-m| \le \left(\sup_{z\in \mathrm{B}(v,|v'-v|)} |\nabla m|\,(z)\right)\,|v'-v|,$$

with

$$|v'-v| = |v-v_*|/2\,\sin(\theta/2) \le \frac{1}{2\sqrt{2}}\,|v-v_*|\ \ \sin\theta.$$

Then, we use the fact

$$\begin{aligned}
\sup_{z\in \mathrm{B}(v,|v'-v|)} |\nabla m|\,(z) &\le k\,2^{k-1}\,\left(\langle v\rangle^{k-2} + \langle v-v'\rangle^{k-1}\right) \\
&\le k\,2^{2(k-1)}\,\left(\langle v\rangle^{k-2} + \langle v_*\rangle^{k-1}\right),
\end{aligned}$$

which implies that

$$|m'-m| \le C_k\,|v-v_*|\,\sin\theta\,\left(\langle v\rangle^{k-1} + \langle v_*\rangle^{k-1}\right), \tag{2.19}$$

for some constant $C_k > 0$ depending on k.

Remark 2.2.31. *We here point out that this kind of estimate does not hold in the case of a stretched exponential weight. Indeed, taking the gradient of a stretched exponential function, there is not anymore a gain in the degree as in the case of a polynomial function.*

We finally obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{L}^1_\delta(h)\,\mathrm{sign}(h)\, m\, dv \\
&\leq C_k \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\sin\theta\,\mu_*\,|v-v_*|^{\gamma+1}\left(\langle v\rangle^{k-1}+\langle v_*\rangle^{k-1}\right)|h|\,d\sigma\,dv_*\,dv \\
&\leq C_k \int_0^{\pi/2} b_\delta(\cos\theta)\sin^2(\theta)d\theta\int_0^{2\pi}d\phi\int_{\mathbb{R}^3\times\mathbb{R}^3}\mu_*|v-v_*|^{\gamma+1}\left(\langle v\rangle^{k-1}+\langle v_*\rangle^{k-1}\right)|h|dv_*\,dv \\
&\leq C_k\ \kappa_\delta \int_{\mathbb{R}^3} |h|\,\langle v\rangle^\gamma\, m\, dv,
\end{aligned} \tag{2.20}$$

where we used spherical coordinates to obtain the second inequality and (2.18) to obtain the last one.

We now deal with $\mathcal{L}^2_\delta$. We split it into two parts:

$$\begin{aligned}
\mathcal{L}^2_\delta h &= \int_{\mathbb{R}^3\times\mathbb{S}^2}\left[h'_*\,\mu' - h_*\,\mu\right]\, b_\delta(\cos\theta)|v-v_*|^\gamma\, d\sigma\, dv_* \\
&= \int_{\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\, h'_*\,\left[\mu'-\mu\right]\, d\sigma\, dv_* + \int_{\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\left[h'_*-h_*\right] d\sigma dv_*\,\mu \\
&=: \mathcal{L}^{2,1}_\delta h + \mathcal{L}^{2,2}_\delta h.
\end{aligned}$$

Concerning $\mathcal{L}^{2,2}_\delta$, we use the cancellation lemma [1, Lemma 1]. It implies that

$$\mathcal{L}^{2,2}_\delta h = (S_\delta * h)\ \mu$$

with

$$\begin{aligned}
S_\delta(z) &= 2\pi\int_0^{\pi/2}\sin\theta\, b_\delta(\cos\theta)\left(\frac{|z|^\gamma}{\cos^{\gamma+3}(\theta/2)} - |z|^\gamma\right) d\theta \\
&= 2\pi\,|z|^\gamma\int_0^{\pi/2}\sin\theta\, b_\delta(\cos\theta)\,\frac{1-\cos^{\gamma+3}(\theta/2)}{\cos^{\gamma+3}(\theta/2)}\, d\theta \\
&= 2\pi\,|z|^\gamma\int_0^{\delta}\sin\theta\, b(\cos\theta)\,\frac{1-\cos^{\gamma+3}(\theta/2)}{\cos^{\gamma+3}(\theta/2)}\, d\theta \\
&\leq C\,|z|^\gamma\int_0^{\delta}\sin\theta\, b(\cos\theta)\,\theta^2\, d\theta \\
&\leq C\,\delta^{2-2s}\,|z|^\gamma,
\end{aligned}$$

where the next-to-last inequality comes from the fact that $\frac{1-\cos^{\gamma+3}(\theta/2)}{cos^{\gamma+3}(\theta/2)} \sim \frac{\gamma+3}{2}\,\theta^2$ as θ goes to 0. We hence deduce that for any $\theta\in(0,\delta)$, $\frac{1-\cos^{\gamma+3}(\theta/2)}{cos^{\gamma+3}(\theta/2)} \leq C\,\theta^2$ for some $C>0$;

and the last inequality comes from (2.3). We deduce that

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{L}_\delta^{2,2}(h)\,\mathrm{sign}(h)\,m\,dv &\leq \int_{\mathbb{R}^3} |S_\delta * h|\ m\,\mu\,dv \\ &\leq C\,\delta^{2-2s} \int_{\mathbb{R}^3} (|\ \cdot|^\gamma * |h|)\ \mu\,m\,dv \\ &\leq C\,\delta^{2-2s} \int_{\mathbb{R}^3} (|\ \cdot|^\gamma * \mu\,m)\ |h|\,dv \\ &\leq C\,\delta^{2-2s} \int_{\mathbb{R}^3} |h|\ \langle v\rangle^\gamma\,dv.\end{aligned} \tag{2.21}$$

We now deal with $\mathcal{L}_\delta^{2,1}$. To do that, we introduce the notation $M := \sqrt{\mu}$ and write that $\mu' - \mu = (M' - M)(M' + M)$, which implies

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{L}_\delta^{2,1}(h)\,\mathrm{sign}(h)\,m\,dv &\leq \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h'_*|\,|M'-M|\,(M'+M)\,m\,d\sigma\,dv_*\,dv \\ &\leq \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h'_*|\,|M'-M|\,M'\,|m'-m|\,d\sigma\,dv_*\,dv \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h'_*|\,|M'-M|\,M'\,m'\,d\sigma\,dv_*\,dv \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h'_*|\,|M'-M|\,M\,m\,d\sigma\,dv_*\,dv.\end{aligned}$$

We now perform the pre-post collisional change of variables, which gives us:

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{L}_\delta^{2,1} h\,\mathrm{sign}(h)\,m\,dv &\leq \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h_*|\,|M'-M|\,M\,|m'-m|\,d\sigma\,dv_*\,dv \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h_*|\,|M'-M|\,M\,m\,d\sigma\,dv_*\,dv \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|h_*|\,|M'-M|\,M'\,m'\,d\sigma\,dv_*\,dv \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

For the term I_1, we use the fact that M is bounded and the estimate (2.19) on $|m'-m|$:

$$\begin{aligned}I_1 &\leq C_k \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\sin\theta\,|v-v_*|^{\gamma+1}\,|h_*|\,M\,\left(\langle v\rangle^{k-1} + \langle v_*\rangle^{k-1}\right)\,d\sigma\,dv_*\,dv \\ &\leq C_k\,\kappa_\delta \int_{\mathbb{R}^3} |h|\ \langle v\rangle^\gamma\,m\,dv.\end{aligned} \tag{2.22}$$

The term I_2 is treated using that M is Lipschitz continuous, we obtain:

$$I_2 \leq C\,\kappa_\delta \int_{\mathbb{R}^3} |h|\ \langle v\rangle^{\gamma+1}\,dv. \tag{2.23}$$

To treat I_3, we first estimate the integral

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,|M'-M|\,M'\,m'\,d\sigma\,dv =: J(v_*) = J.$$

Using the fact that M is Lipschitz continuous, we have

$$J \leq C \int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\, \sin(\theta/2)\, |v-v_*|^{\gamma+1}\, M'\, m'\, d\sigma\, dv.$$

Then, for each σ, with v_* still fixed, we perform the change of variables $v \to v'$. This change of variables is well-defined on the set $\{\cos\theta > 0\}$. Its Jacobian determinant is

$$\left|\ \frac{dv'}{dv}\ \right| = \frac{1}{8}(1+\kappa\cdot\sigma) = \frac{(\kappa'\cdot\sigma)^2}{4},$$

where $\kappa = (v-v_*)/|v-v_*|$ and $\kappa' = (v'-v_*)/|v'-v_*|$. We have $\kappa'\cdot\sigma = \cos(\theta/2) \geq 1/\sqrt{2}$. The inverse transformation $v' \to \psi_\sigma(v') = v$ is then defined accordingly. Using the fact that

$$\cos\theta = \kappa\cdot\sigma = 2(\kappa'\cdot\sigma)^2 - 1 \quad \text{and} \quad \sin(\theta/2) = \sqrt{1-\cos^2(\theta/2)} = \sqrt{1-(\kappa'\cdot\sigma)^2},$$

we obtain

$$\begin{aligned}
J &\leq C \int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta(2(\kappa'\cdot\sigma)^2-1)\, \sqrt{1-(\kappa'\cdot\sigma)^2}\, |\psi_\sigma(v')-v_*|^{\gamma+1}\ M(v')\, m(v')\, dv\, d\sigma \\
&\leq C \int_{\kappa'\cdot\sigma\geq 1/\sqrt{2}} b_\delta(2(\kappa'\cdot\sigma)^2-1)\sqrt{1-(\kappa'\cdot\sigma)^2}|\psi_\sigma(v')-v_*|^{\gamma+1}\ M(v')m(v')\frac{1}{(\kappa'\cdot\sigma)^2}dv'd\sigma \\
&\leq C \int_{\kappa\cdot\sigma\geq 1/\sqrt{2}} b_\delta(2(\kappa\cdot\sigma)^2-1)\, \sqrt{1-(\kappa\cdot\sigma)^2}\, |\psi_\sigma(v)-v_*|^{\gamma+1}\ M(v)\, m(v)\, \frac{1}{(\kappa\cdot\sigma)^2}\, dv\, d\sigma.
\end{aligned}$$

We now use the fact that $|\psi_\sigma(v)-v_*| = |v-v_*|/(\kappa\cdot\sigma)$. We deduce that

$$\begin{aligned}
J &\leq C \int_{\kappa\cdot\sigma\geq 1/\sqrt{2}} b_\delta(2(\kappa\cdot\sigma)^2-1)\, \sqrt{1-(\kappa\cdot\sigma)^2}\, |v-v_*|^{\gamma+1}\ M(v)\, m(v)\, \frac{1}{(\kappa\cdot\sigma)^{\gamma+3}}\, dv\, d\sigma \\
&\leq C \int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta(2(\kappa\cdot\sigma)^2-1)\, \sqrt{1-(\kappa\cdot\sigma)^2}\, |v-v_*|^{\gamma+1}\ M(v)\, m(v)\, dv\, d\sigma
\end{aligned}$$

where we used the fact that $\kappa\cdot\sigma \geq 1/\sqrt{2}$ to bound from above $1/(\kappa\cdot\sigma)^{\gamma+3}$. Using the equalities

$$\cos(2\theta) = 2(\kappa\cdot\sigma)^2 - 1 \quad \text{and} \quad \sin\theta = \sqrt{1-(\kappa\cdot\sigma)^2},$$

we obtain

$$\begin{aligned}
J &\leq C \int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos(2\theta))\, \sin\theta\, |v-v_*|^{\gamma+1}\, M\, m\, dv\, d\sigma \\
&\leq C \int_{\mathbb{S}^2} b_\delta(\cos(2\theta))\, \sin\theta\, d\sigma\ \int_{\mathbb{R}^3} \langle v\rangle^{\gamma+1}\, M\, m\, dv\, \langle v_*\rangle^{\gamma+1} \\
&\leq C\, \kappa_\delta\, \langle v_*\rangle^{\gamma+1},
\end{aligned}$$

Using this last estimate, we can conclude that

$$I_3 \leq C\, \kappa_\delta \int_{\mathbb{R}^3} |h|\ \langle v\rangle^{\gamma+1}\, dv. \tag{2.24}$$

Gathering estimates (2.20), (2.21), (2.22), (2.23) and (2.24), we can conclude that (2.17) holds. □

We now want to deal with the part $\mathcal{B}^c_{\delta,\varepsilon} - \nu_\delta$. To do that, we shall review a classical tool in the Boltzmann theory, a version of the Povzner lemma (see [99, 17, 72, 18]). The version stated here is a consequence of the proof of Lemma 2.2 from [72].

Lemma 2.2.32. *For any $k > 2$,*

$$\forall\, v,\, v_* \in \mathbb{R}^3, \quad \int_{\mathbb{S}^2} \left[\langle v'_* \rangle^k + \langle v' \rangle^k - \langle v_* \rangle^k - \langle v \rangle^k \right] b^c_\delta(\cos\theta)\, d\sigma \\ \le C_k \left(\langle v_* \rangle^{k-1} \, \langle v \rangle + \langle v \rangle^{k-1} \langle v_* \rangle \right) - C'_k \, |v|^k$$

for some constants C_k, $C'_k > 0$ depending on k.

Proof. If we adapt the proof of Lemma 2.2 from [72] taking $\psi = \langle \cdot \rangle$, we obtain

$$\int_{\mathbb{S}^2} \left[\langle v'_* \rangle^k + \langle v' \rangle^k - \langle v_* \rangle^k - \langle v \rangle^k \right] b^c_\delta(\cos\theta)\, d\sigma \\ \le C_k \left(\int_0^{\pi/2} b^c_\delta(\cos\theta) \sin^2(\theta)\, d\theta \right) \left(\langle v_* \rangle^{k-1} \, \langle v \rangle + \langle v \rangle^{k-1} \langle v_* \rangle \right) - C'_{k,\delta} \, |v|^k$$

with $C'_{k,\delta} \xrightarrow[\delta \to 0]{} +\infty$ and $C'_{k,\delta} \ge C'_k > 0$ for any $\delta \in (0,1)$. We then conclude using (2.3) which implies that

$$\int_0^{\pi/2} b^c_\delta(\cos\theta) \sin^2(\theta)\, d\theta \approx \left(\frac{\pi}{2} \right)^{1-2s} - \delta^{1-2s} \le C$$

for any $\delta \in (0,1)$. □

We can now prove the following estimate on $\mathcal{B}^c_{\delta,\varepsilon} - \nu_\delta$.

Lemma 2.2.33. *For any $k > 2$, for any $\varepsilon \in (0,1)$ and for $\delta \in (0,1)$ small enough, we have the following estimate : for any $h \in L^1(\langle v \rangle^\gamma m)$,*

$$\int_{\mathbb{R}^3} \mathcal{B}^c_{\delta,\varepsilon}(h) \,\mathrm{sign}(h)\, m\, dv - \int_{\mathbb{R}^3} \nu_\delta \, |h| \, m \, dv \le (\Lambda_{k,\delta}(\varepsilon) - \lambda_k) \, \|h\|_{L^1(\langle v \rangle^\gamma m)} \tag{2.25}$$

where $\lambda_k > 0$ is a constant depending on k and $\Lambda_{k,\delta}(\varepsilon)$ is a constant depending on k and δ which tends to 0 as ε goes to 0 when k and δ are fixed.

Proof. We compute

$$\|\mathcal{B}^c_{\delta,\varepsilon} h\|_{L^1(m)} \le \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (1 - \Theta_\varepsilon) \, B^c_\delta(v - v_*, \sigma) \left[\, \mu'_* |h'| + \mu' |h'_*| + \mu |h_*| \, \right] m \, d\sigma \, dv_* \, dv.$$

We first bound from above the truncation function $(1 - \Theta_\varepsilon)$:

$$\|\mathcal{B}^c_{\delta,\varepsilon} h\|_{L^1(m)} \le \int_{\{|\cos\theta| \in [1-\varepsilon, 1]\}} B^c_\delta(v - v_*, \sigma) \, \mu_* |h| \, \left(m' + m'_* + m_* \right) d\sigma \, dv_* \, dv \\ + \int_{\{|v - v_*| \, \le \varepsilon\}} B^c_\delta(v - v_*, \sigma) \, \mu_* |h| \, \left(m' + m'_* + m_* \right) d\sigma \, dv_* \, dv \\ + \int_{\{|v| \, \ge \varepsilon^{-1} \text{ or } \, |v - v_*| \, \ge \varepsilon^{-1}\}} B^c_\delta(v - v_*, \sigma) \left[\, \mu'_* |h'| + \mu' |h'_*| + \mu |h_*| \, \right] m \, d\sigma \, dv_* \, dv,$$

where the pre-post collisional change of variables has been used in the two first terms. We obtain that $\|\mathcal{B}^c_{\delta,\varepsilon} h\|_{L^1(m)}$ is bounded from above by

$$
\begin{aligned}
&C_k \left(\int_{\{|\cos\theta| \in [1-\varepsilon,1]\}} \mathbb{1}_{\theta \geq \delta}\, b(\cos\theta)\, d\sigma + K_\delta\, \varepsilon^\gamma \right) \int_{\mathbb{R}^3\times\mathbb{R}^3} \mu_* \langle v_* \rangle^{\gamma+k} \, |h| \, \langle v \rangle^{\gamma+k} \;\; dv_*\, dv \\
&\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_{\varepsilon^{-1}} B^c_\delta(v-v_*,\sigma) \left[\; \mu'_*|h'| + \mu'|h'_*| + \mu|h_*| \;\right] \, m\, d\sigma\, dv_*\, dv \\
&=: J_1 + J_2
\end{aligned}
\tag{2.26}
$$

where $\chi_{\varepsilon^{-1}}$ is the characteristic function of the set

$$
\left\{ \sqrt{|v|^2 + |v_*|^2} \;\; \geq \varepsilon^{-1} \text{ or } |v - v_*| \;\; \geq \varepsilon^{-1} \right\}.
$$

The first term of the right hand side of (2.26) is easily controlled as

$$
J_1 \leq C_k\, C_\delta\, \varepsilon^\gamma \, \|h\|_{L^1(\langle v \rangle^\gamma m)}. \tag{2.27}
$$

As far as the second term in (2.26) is concerned, we write

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_{\varepsilon^{-1}} B^c_\delta(v-v_*,\sigma) \left[\; \mu'_*|h'| + \mu'|h'_*| + \mu|h_*| \;\right] \, m\, d\sigma\, dv_*\, dv \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \chi_{\varepsilon^{-1}} B^c_\delta(v-v_*,\sigma) \left[\; \mu'_*|h'| + \mu'|h'_*| - \mu_*|h| - \mu|h_*| \;\right] \, m\, d\sigma\, dv_*\, dv \\
&\quad + K_\delta \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h| \;\; |v-v_*|^\gamma \, m\, dv_*\, dv \\
&\quad + 2\, K_\delta \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu |h_*| \;\; |v-v_*|^\gamma \, m\, dv_*\, dv \\
&=: T_1 + T_2 + T_3.
\end{aligned}
$$

We notice that the characteristic function $\chi_{\varepsilon^{-1}}$ is invariant under the usual pre-post collisional change of variables as it only depends on the kinetic energy and momentum. We hence bound the term T_1 thanks to Lemma 2.2.32:

$$
\begin{aligned}
T_1 &\leq \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h|\, |v-v_*|^\gamma \int_{\mathbb{S}^2} \left(\langle v'_* \rangle^k + \langle v' \rangle^k - \langle v_* \rangle^k - \langle v \rangle^k \right) b^c_\delta(\cos\theta)\, d\sigma\, dv_*\, dv \\
&\leq C_k \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h|\, |v-v_*|^\gamma \, \left(\langle v \rangle^{k-1} \langle v_* \rangle + \langle v \rangle \langle v_* \rangle^{k-1} \right) dv_*\, dv \\
&\quad - C'_k \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h|\, |v-v_*|^\gamma \, |v|^k \, dv_*\, dv \\
&\leq C_k \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h|\, |v-v_*|^\gamma \, \left(\langle v \rangle^{k-1} \langle v_* \rangle + \langle v \rangle \langle v_* \rangle^{k-1} \right) dv_*\, dv \\
&\quad + C'_k \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h|\, |v-v_*|^\gamma \, dv_*\, dv \\
&\quad - C'_k\, 2^{1-k/2} \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\, \mu_* |h|\, |v-v_*|^\gamma \, \langle v \rangle^k \, dv_*\, dv \\
&=: T_{11} + T_{12} + T_{13}.
\end{aligned}
$$

We treat together the terms T_{11}, T_{12} and T_3 using the following inequality:

$$\chi_{\varepsilon^{-1}}(v,v_*) \le \mathbb{1}_{\{|v|\ge \varepsilon^{-1}/2\}} + \mathbb{1}_{\{|v_*|\ge \varepsilon^{-1}/2\}} \le 2\,\varepsilon(|v|+|v_*|).$$

We obtain:

$$\begin{aligned}
&T_{11}+T_{12}+T_3 \\
&\le \varepsilon\, C_k \int_{\mathbb{R}^3\times\mathbb{R}^3} (|v|+|v_*|)\,\mu_*|h|\,|v-v_*|^\gamma \, \left(\langle v\rangle^{k-1}\langle v_*\rangle + \langle v\rangle\langle v_*\rangle^{k-1}\right) dv_*\,dv \\
&\quad + \varepsilon\, C_k' \int_{\mathbb{R}^3\times\mathbb{R}^3} (|v|+|v_*|)\,\mu_*|h|\,|v-v_*|^\gamma \, dv_*\,dv \\
&\quad + \varepsilon\, K_\delta \int_{\mathbb{R}^3\times\mathbb{R}^3} (|v|+|v_*|)\,\mu|h_*|\;\;|v-v_*|^\gamma \, m\, dv_*\,dv \\
&\le \varepsilon\, C_k\, C_\delta\, \|h\|_{L^1(\langle v\rangle^\gamma m)}.
\end{aligned} \tag{2.28}$$

Gathering (2.27) and (2.28), we conclude that

$$J_1+T_{11}+T_{12}+T_3 \le C_k\,C_\delta\,(\varepsilon+\varepsilon^\gamma)\,\|h\|_{L^1(\langle v\rangle^\gamma m)} =: \Lambda_{k,\delta}(\varepsilon)\,\|h\|_{L^1(\langle v\rangle^\gamma m)}. \tag{2.29}$$

We now put together the terms T_{13}, T_2 and the term coming from ν_δ, their sum is bounded from above by

$$-K_\delta \int_{\mathbb{R}^3\times\mathbb{R}^3} (1-\chi_{\varepsilon^{-1}})\mu_*|v-v_*|^\gamma |h| m dv_* dv - C_k' 2^{1-k/2} \int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_{\varepsilon^{-1}}\mu_*|v-v_*|^\gamma|h| m dv_* dv.$$

Since $K_\delta \to \infty$ as $\delta \to 0$, we can take δ small enough so that $K_\delta \ge C_k'\,2^{1-k/2}$, we obtain the following bound:

$$-\,C_k'\,2^{1-k/2}\int_{\mathbb{R}^3} (\mu * |\cdot|^\gamma)\,|h|\,m\,dv \le -\lambda_k\,\|h\|_{L^1(\langle v\rangle^\gamma m)}. \tag{2.30}$$

Combining the bounds obtained in (2.29) and (2.30), we can conclude that (2.25) holds, which concludes the proof. □

We can now prove the dissipativity properties of $\mathcal{B}_{\delta,\varepsilon} = \mathcal{L}_\delta + \mathcal{B}^c_{\delta,\varepsilon} - \nu_\delta$.

Lemma 2.2.34. *Let us consider $a \in (-\lambda_k, 0)$ where λ_k is defined in Lemma 2.2.33. For $\delta > 0$ and $\varepsilon > 0$ small enough, $\mathcal{B}_{\delta,\varepsilon} - a$ is dissipative in $L^1(m)$, namely*

$$\forall\, t\ge 0, \quad \|S_{\mathcal{B}_{\delta,\varepsilon}}(t)\|_{L^1(m)\to L^1(m)} \le e^{at}.$$

Proof. Gathering results coming from lemmas 2.2.29 and 2.2.33, we obtain

$$\int_{\mathbb{R}^3} \mathcal{B}_{\delta,\epsilon}(h)\,\mathrm{sign}(h)\,m\,dv \le \int_{\mathbb{R}^3} (\varphi_k(\delta)+\Lambda_{k,\delta}(\varepsilon)-\lambda_k)\,|h|\,\langle v\rangle^\gamma\,m\,dv$$

We first take δ small enough so that $\varphi_k(\delta) \le (a+\lambda_k)/2$. We then chose ε small enough so that $\Lambda_{k,\delta}(\varepsilon) \le (a+\lambda_k)/2$. With this choice of δ and ε, we have the following inequality:

$$\varphi_k(\delta)+\Lambda_{k,\delta}(\varepsilon)-\lambda_k \le a.$$

It implies that

$$\int_{\mathbb{R}^3} \mathcal{B}_{\delta,\epsilon}(h)\,\mathrm{sign}(h)\,m\,dv \le a\|h\|_{L^1(\langle v\rangle^\gamma m)},$$

which concludes the proof. □

Regularization properties

We first state a regularity estimate on the truncated operator $\mathcal{A}_{\delta,\varepsilon}$ which comes from [51, Lemma 4.16].

Lemma 2.2.35. *The operator $\mathcal{A}_{\delta,\varepsilon}$ maps $L^1(\langle v\rangle)$ into L^2 functions with compact support. In particular, we can deduce that $\mathcal{A}_{\delta,\varepsilon} \in \mathscr{B}\left(L^2\left(\mu^{-1/2}\right)\right)$ and $\mathcal{A}_{\delta,\varepsilon} \in \mathscr{B}\left(L^1(m)\right)$.*

We now study the regularization properties of $T(t) := \mathcal{A}_{\delta,\varepsilon} S_{\mathcal{B}_{\delta,\varepsilon}}(t)$.

Lemma 2.2.36. *Consider $a \in (-\lambda_k, 0)$. For a choice of δ, ε such that the conclusion of Lemma 2.2.34 holds, there exists a constant $C > 0$ such that*

$$\|T(t)h\|_{L^2(\mu^{-1/2})} \leq C\, e^{at}\, \|h\|_{L^1(m)}.$$

Proof. We here use Lemma 2.2.35. We introduce a constant $R > 0$ such that for any h in $L^1(\langle v\rangle)$, $\operatorname{supp}(\mathcal{A}h) \subset B(0,R)$. We then compute

$$\begin{aligned}\|T(t)h\|_{L^2(\mu^{-1/2})} &\leq C\left(\int_{B(0,R)} (T(t)h)^2\, dv\right)^{1/2} \leq C\, \|S_{\mathcal{B}_{\delta,\varepsilon}}(t)h\|_{L^1(\langle v\rangle)} \\ &\leq C\, \|S_{\mathcal{B}_{\delta,\varepsilon}}(t)h\|_{L^1(m)} \leq C\, e^{at}\, \|h\|_{L^1(m)},\end{aligned}$$

where the last inequality comes from Lemma 2.2.34. □

2.2.4 Spectral gap in $L^1(\langle v\rangle^k)$

The abstract theorem

Let us now present an enlargement of the functional space of a quantitative spectral mapping theorem (in the sense of semigroup decay estimate). The aim is to enlarge the space where the decay estimate on the semigroup holds. The version stated here comes from [51, Theorem 2.13].

Theorem 2.2.37. *Let E, $\mathcal{E}$ be two Banach spaces such that $E \subset \mathcal{E}$ with dense and continuous embedding, and consider $L \in \mathscr{C}(E)$, $\mathcal{L} \in \mathscr{C}(\mathcal{E})$ with $\mathcal{L}_{|E} = L$ and $a \in \mathbb{R}$. We assume:*

(1) *L generates a semigroup $S_L(t)$ and*

$$\Sigma(L) \cap \Delta_a = \{\xi\} \subset \Sigma_d(L)$$

for some $\xi \in \mathbb{C}$ and $L - a$ is dissipative on $\mathrm{R}\left(\mathrm{Id} - \Pi_{L,\xi}\right)$.

(2) *There exist $\mathcal{A}$, $\mathcal{B} \in \mathscr{C}(\mathcal{E})$ such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ (with corresponding restrictions A and B on E) and a constant $C_a > 0$ so that*

- **(i)** *$\mathcal{B} - a$ is dissipative on $\mathcal{E}$,*
- **(ii)** *$A \in \mathscr{B}(E)$ and $\mathcal{A} \in \mathscr{B}(\mathcal{E})$,*

(iii) *$T(t) := \mathcal{A}S_{\mathcal{B}}(t)$ satisfies*

$$\forall\, t \geq 0, \quad \|T(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq C_a\, e^{at}.$$

Then the following estimate on the semigroup holds:

$$\forall\, a' > a, \forall\, t \geq 0, \quad \|S_{\mathcal{L}}(t) - S_L(t)\Pi_{\mathcal{L},\xi}\|_{\mathscr{B}(\mathcal{E})} \leq C_{a'} e^{a't}$$

where $C_{a'} > 0$ is an explicit constant depending on the constants from the assumptions.

Proof of Theorem 2.1.27

The conclusion of Theorem 2.1.27 is a direct consequence of Theorem 2.2.37. Indeed, denoting $E = L^2(\mu^{-1/2})$ and $\mathcal{E} = L^1(m)$, assumption (1) is nothing but Proposition 2.2.28, assumption (2)-(i) comes from Lemma 2.2.34, (2)-(ii) from Lemma 2.2.35 and (2)-(iii) from Lemma 2.2.36. We can conclude that estimate (2.15) holds.

2.3 The nonlinear equation

We first establish bilinear estimates on the collisional operator and we then prove our main result: Theorem 2.1.24.

2.3.1 The bilinear estimates

Proposition 2.3.38. *Let B satisfying (2.2), (2.3) and (2.4). Then*

$$\|Q(h,h)\|_{L^1(m)} \leq C \left(\|h\|_{L^1(\langle v\rangle^\gamma m)} \|h\|_{L^1(m)} + \|h\|_{L^1(\langle v\rangle^{\gamma+1})} \|h\|_{W^{1,1}(\langle v\rangle^{\gamma+1} m)} \right)$$

for some $C > 0$.

Proof. We split $Q(h,h)$ into two parts and we use the pre-post collisional change of variables for the second one, we obtain

$$\begin{aligned}
\|Q(h,h)\|_{L^1(m)} &= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\ \left((h'_* - h_*)h + (h'-h)h'_*\right)\ d\sigma\, dv_* \right| m\, dv \\
&\leq \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\, (h'_* - h_*)\, d\sigma\, dv_* \right| |h|\, m\, dv \\
&\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\, |h'-h|\, |h'_*|\ \ m\, d\sigma\, dv_*\, dv \\
&\leq \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\, (h'_* - h_*)\, d\sigma\, dv_* \right| |h|\, m\, dv \\
&\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\, |h'-h|\, |h_*|\ \ m'\, d\sigma\, dv_*\, dv \\
&=: T_1 + T_2.
\end{aligned}$$

We first deal with T_1 using the cancellation lemma [1, Lemma 1]:

$$T_1 = \int_{\mathbb{R}^3} |S * h| \, |h| \, m \, dv$$

with

$$\begin{aligned} S(z) &= 2\pi \int_0^{\pi/2} \sin\theta \, b(\cos\theta) \left(\frac{|z|^\gamma}{\cos^{\gamma+3}(\theta/2)} - |z|^\gamma \right) d\theta \\ &= 2\pi \, |z|^\gamma \int_0^{\pi/2} \sin\theta \, b(\cos\theta) \, \frac{1 - \cos^{\gamma+3}(\theta/2)}{\cos^{\gamma+3}(\theta/2)} \, d\theta \\ &\le C \, |z|^\gamma. \end{aligned}$$

We deduce that

$$T_1 \le C \, \|h\|_{L^1(\langle v \rangle^\gamma)} \|h\|_{L^1(\langle v \rangle^\gamma m)}. \tag{2.31}$$

We now treat the term T_2 which is splitted into two parts:

$$\begin{aligned} T_2 &= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \, |h' m' - h m'| \, |h_*| \, d\sigma \, dv_* \, dv \\ &\le \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \, |h' m' - h m| \, |h_*| \, d\sigma \, dv_* \, dv \\ &\quad + \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \, |m' - m| \, |h| \, |h_*| \, d\sigma \, dv_* \, dv \\ &=: T_{21} + T_{22}. \end{aligned}$$

Concerning T_{21}, we have to estimate

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cos\theta) \, |v - v_*|^\gamma \, |h' m' - h m| \, dv \, d\sigma =: J(v_*) = J.$$

To do that, we use Taylor formula denoting $\overline{v}_u := (1-u)v + uv'$ for any $u \in [0,1]$, which allows us to estimate $|h'm' - hm|$:

$$\begin{aligned} |h' m' - h m| &= \left| \int_0^1 \nabla(hm)(\overline{v}_u) \cdot (v - v') \, du \right| \\ &\le \int_0^1 |\nabla(hm)(\overline{v}_u)| \, |v - v_*| \, \sin(\theta/2) \, du. \end{aligned}$$

It implies the following inequality on J:

$$J \le C \int_{\mathbb{R}^3 \times \mathbb{S}^2 \times [0,1]} b(\cos\theta) \, \sin(\theta) \, |v - v_*|^{\gamma+1} \, |\nabla(hm)(\overline{v}_u)| \, du \, d\sigma \, dv.$$

Moreover, if $v \neq v_*$, we have the following equality:

$$|v - v_*| = \frac{1}{\left| \left(1 - \frac{u}{2}\right) \kappa + \frac{u}{2} \sigma \right|} \, |\overline{v}_u - v_*|.$$

Using the fact that $0 \le \langle \kappa, \sigma \rangle \le 1$, one can show that for any $u \in [0,1]$,

$$\left| \left(1 - \frac{u}{2}\right) \kappa + \frac{u}{2}\, \sigma \right| \ge \frac{1}{\sqrt{2}}.$$

We can thus deduce that for any $u \in [0,1]$, we have $|v - v_*| \le C |\overline{v}_u - v_*|$ for some $C > 0$, which implies

$$J \le C \int_{\mathbb{R}^3 \times \mathbb{S}^2 \times [0,1]} b(\cos\theta)\, \sin(\theta)\, |\overline{v}_u - v_*|^{\gamma+1}\, |\nabla(hm)(\overline{v}_u)|\, du\, d\sigma\, dv.$$

For u, v_* and σ fixed, we now perform the change of variables $v \to \overline{v}_u$. Its Jacobian determinant is

$$\left| \frac{d\overline{v}_u}{dv} \right| = \left(1 - \frac{u}{2}\right)^2 \left(1 - \frac{u}{2} + \frac{u}{2} \langle \kappa, \sigma \rangle \right) \ge \left(1 - \frac{u}{2}\right)^3 \ge \frac{1}{8}$$

since $\langle \kappa, \sigma \rangle \ge 0$. Gathering all the previous estimates, we obtain

$$J \le C \int_{\mathbb{S}^2} b(\cos\theta)\, \sin(\theta)\, d\sigma \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1}\, |\nabla(hm)(v)|\, dv.$$

We thus obtain :

$$T_{21} \le C\, \|h\|_{L^1(\langle v \rangle^{\gamma+1})}\, \|h\|_{W^{1,1}(\langle v \rangle^{\gamma+1} m)}. \tag{2.32}$$

Let us finally deal with T_{22}. We here use the inequality (2.19):

$$\begin{aligned} T_{22} &\le C \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma)\, |h|\, |h_*| \left(\langle v \rangle^{k-1} + \langle v_* \rangle^{k-1} \right) |v' - v|\, d\sigma\, dv_*\, dv \\ &\le C \int_{\mathbb{S}^2} b(\cos\theta)\, \sin(\theta)\, d\sigma \int_{\mathbb{R}^3 \times \mathbb{R}^3} |h|\, |h_*| \left(\langle v \rangle^{k-1} + \langle v_* \rangle^{k-1} \right) |v - v_*|^{\gamma+1}\, dv_*\, dv \\ &\le C\, \|h\|_{L^1(\langle v \rangle^{\gamma} m)}\, \|h\|_{L^1(\langle v \rangle^{\gamma+1})}. \end{aligned} \tag{2.33}$$

Inequalities (2.31), (2.32) and (2.33) together yield the result. □

We now recall a classical result from interpolation theory (see for example Lemma B.1 from [66]).

Lemma 2.3.39. *For any s, s^*, q, $q^* \in \mathbb{Z}$ with $s \ge s^*$, $q \ge q^*$ and any $\theta \in (0,1)$, there exists $C > 0$ such that for any $h \in W^{s^{**},1}(\langle v \rangle^{q^{**}})$, we have*

$$\|h\|_{W^{s,1}(\langle v \rangle^{q})} \le C\, \|h\|^{1-\theta}_{W^{s^*,1}(\langle v \rangle^{q^*})}\, \|h\|^{\theta}_{W^{s^{**},1}(\langle v \rangle^{q^{**}})}$$

with s^{**}, $q^{**} \in \mathbb{Z}$ such that $s = (1-\theta)s^* + \theta s^{**}$ and $q = (1-\theta)q^* + \theta q^{**}$.

It allows us to prove the following corollary which is going to be useful in the proof of our main theorem.

Corollary 2.3.40. *Let B satisfying (2.2), (2.3) and (2.4). Then*

$$\|Q(h,h)\|_{L^1(m)} \le C\left(\|h\|^{3/2}_{L^1(m)}\|h\|^{1/2}_{L^1(\langle v\rangle^{2\gamma}m)} + \|h\|^{3/2}_{L^1(m)}\|h\|^{1/2}_{H^4(\langle v\rangle^{4\gamma+k+6})}\right).$$

Proof. On the one hand, using Lemma 2.3.39, we obtain:

$$\|h\|_{L^1(\langle v\rangle^{\gamma}m)} \le \|h\|^{1/2}_{L^1(\langle v\rangle^{2\gamma}m)}\|h\|^{1/2}_{L^1(m)}.$$

On the other hand, again using twice Lemma 2.3.39, we obtain

$$\begin{aligned}\|h\|_{L^1(\langle v\rangle^{\gamma+1})}\|h\|_{W^{1,1}(\langle v\rangle^{\gamma+1}m)} &\le \|h\|^2_{W^{1,1}(\langle v\rangle^{\gamma+k+1})}\\ &\le C\|h\|_{L^1(m)}\ \|h\|_{W^{2,1}(\langle v\rangle^{2\gamma+k+2})}\\ &\le C\|h\|^{3/2}_{L^1(m)}\|h\|^{1/2}_{W^{4,1}(\langle v\rangle^{4\gamma+k+4})}.\end{aligned}$$

To conclude we use that for any $q\in\mathbb{N}$, we can show using Hölder inequality that

$$\|h\|_{L^1(\langle v\rangle^{q})} \le C\|h\|_{L^2(\langle v\rangle^{q+2})}.$$

□

2.3.2 Proof of Theorem 2.1.24

Let $f_0 = \mu + h_0$ and consider the equation

$$\partial_t h_t = \mathcal{L}h_t + Q(h_t,h_t), \quad h(t=0) = h_0. \tag{2.34}$$

Let us notice that for any $t\ge 0$, we have $\Pi\, h_t = 0$. Indeed, f_0 has same mass, momentum and energy as μ, it implies that $\Pi\, h_0 = 0$ and these quantities are conserved by the equation.

We now state a nonlinear stability theorem which is the third key point (with Theorems 2.1.26 and 2.1.27) in the proof of Theorem 2.1.24.

Theorem 2.3.41. *Consider a solution h_t to (2.34) such that*

$$\forall\, t\ge 0, \quad \|h_t\|_{H^4(\langle v\rangle^{4\gamma+k+6})} \le K$$

for some $K>0$. There exists $\eta>0$ such that if moreover

$$\forall\, t\ge 0, \quad \|h_t\|_{L^1(\langle v\rangle^{2\gamma}m)} \le \eta$$

then there exists $C>0$ (depending on K and η) such that

$$\forall\, t\ge 0, \quad \|h_t\|_{L^1(m)} \le C\,e^{-\lambda t}\,\|h_0\|_{L^1(m)}$$

for any positive $\lambda < \min(\lambda_0,\lambda_k)$ (see Theorem 2.1.27).

Proof. We use Duhamel's formula for the solution of (2.34):

$$h_t = S_{\mathcal{L}}(t)\, h_0 + \int_0^t S_{\mathcal{L}}(t-s)\, Q(h_s, h_s)\, ds.$$

We now estimate $\|h_t\|_{L^1(m)}$ thanks to Theorem 2.1.27 and Corollary 2.3.40:

$$\begin{aligned} \|h_t\|_{L^1(m)} \leq e^{-\lambda t} \|h_0\|_{L^1(m)} \\ &+ C \int_0^t e^{-\lambda(t-s)} \Big(\|h_s\|_{L^1(m)}^{1/4} \, \|h_s\|_{H^4(\langle v \rangle^{4\gamma+k+6})}^{1/2} + \|h_s\|_{L^1(\langle v \rangle^{2\gamma} m)}^{3/4} \Big) \|h_s\|_{L^1(m)}^{5/4} \, ds \\ &\leq e^{-\lambda t} \, \|h_0\|_{L^1(m)} + C \int_0^t e^{-\lambda(t-s)} \left(K^{1/2} \eta^{1/4} + \eta^{3/4} \right) \|h_s\|_{L^1(m)}^{5/4} \, ds. \end{aligned}$$

We denote $\eta' := C \left(K^{1/2} \eta^{1/4} + \eta^{3/4} \right)$. We end up with a similar differential inequality as in
[75, Lemma 4.5]. We can then conclude in the same way that

$$\forall\, t \geq 0, \quad \|h_t\|_{L^1(m)} \leq C' e^{-\lambda t} \|h_0\|_{L^1(m)},$$

for some $C' > 0$. □

To conclude the proof of Theorem 2.1.24, we consider $\eta > 0$ defined in Theorem 2.3.41. Using Theorem 2.1.26, we can choose $t_1 > 0$ such that

$$\forall\, t \geq t_1, \quad \|h_t\|_{L^1(m)} = \|f_t - \mu\|_{L^1(m)} \ \leq \eta.$$

Thanks to the properties of a smooth solution, we also have

$$\forall\, t \geq t_1, \quad \|h_t\|_{H^4(\langle v \rangle^{4\gamma+k+6})} \leq \|f_t\|_{H^4(\langle v \rangle^{4\gamma+k+6})} + \|\mu\|_{H^4(\langle v \rangle^{4\gamma+k+6})} \leq K$$

for some $K > 0$. We can hence apply Theorem 2.3.41 to h_t starting from t_1. We finally obtain

$$\forall\, t \geq t_1, \quad \|f_t - \mu\|_{L^1(m)} \leq C' e^{-\lambda t} \|h_{t_1}\|_{L^1(m)} \leq C'' e^{-\lambda t},$$

for some $C'' > 0$. The conclusion of Theorem 2.1.24 is hence established.

Chapter 3

Cauchy problem and exponential stability for the inhomogeneous Landau equation

Résumé. Ce travail traite de l'équation de Landau inhomogène en espace dans le tore dans les cas de potentiels durs, maxwellien et faiblement mous. Nous nous intéressons tout d'abord à l'équation linéarisée pour laquelle nous prouvons des estimations de décroissance exponentielle du semi-groupe associé grâce à la théorie développée dans [51]. Nous revenons ensuite au problème non linéaire pour lequel nous construisons des solutions dans un régime proche de l'équilibre (grâce aux estimations sur le semi-groupe linéarisé) dans des espaces de type L^2 à poids polynomial ou exponentiel, élargissant ainsi l'espace dans lequel un tel résultat avait été obtenu dans [52]. Pour finir, nous prouvons une stabilité exponentielle pour une telle solution avec un taux aussi proche que l'on veut du taux optimal donné par la décroissance du semi-groupe.

Abstract. This work deals with the inhomogeneous Landau equation on the torus in the cases of hard, maxwellian and moderately soft potentials. We first investigate the linearized equation and we prove exponential decay estimates for the associated semigroup using the theory developed in [51]. We then turn to the nonlinear equation and we use the linearized semigroup decay in order to construct solutions in a close-to-equilibrium setting in L^2 spaces with polynomial and exponential weights, we thus largely widen the space in which such a result was obtained in [52]. Finally, we prove an exponential stability for the solution constructed, with a rate as close as we want to the optimal rate given by the semigroup decay.

3.1 Introduction

3.1.1 The model

In this paper, we investigate the Cauchy theory associated to the *spatially inhomogeneous Landau equation*. This equation is a kinetic model in plasma physics that describes the evolution of the density function $F = F(t,x,v)$ in the phase space of position and velocities of the particles. In the torus, the equation is given by, for $F = F(t,x,v) \geq 0$ with $t \in \mathbb{R}^+$, $x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ (that we assume without loss of generality to have volume one) and $v \in \mathbb{R}^3$,

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F,F) \\ F_{|t=0} = F_0 \end{cases} \tag{3.1}$$

where the Landau operator Q is a bilinear operator that takes the form

$$Q(G,F)(v) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v-v_*) \left[G_* \partial_j F - F \partial_j G_*\right] dv_*, \tag{3.2}$$

and we use the convention of summation of repeated indices, and the derivatives are in the velocity variable, i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $g_* = g(v_*)$, $f = f(v)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $\partial_j f = \partial_{v_j} f(v)$, etc.

The matrix a_{ij} is symmetric semi-positive, depends on the interaction between particles and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right). \tag{3.3}$$

We define, see [92], in 3-dimension the following quantities

$$\begin{aligned} & b_i(v) = \partial_j a_{ij}(v) = -2\,|v|^\gamma\, v_i, \\ & c(v) = \partial_{ij} a_{ij}(v) = -2(\gamma+3)\,|v|^\gamma \quad \text{or} \quad c = 8\pi\delta_0 \ \text{ if } \ \gamma = -3. \end{aligned} \tag{3.4}$$

We can rewrite the Landau operator (3.2) in the following way

$$Q(G,F) = (a_{ij} *_v G)\ \partial_{ij} F - (c *_v G) F = \nabla_v \cdot \{(a *_v g) \nabla_v f - (b *_v g) f\}. \tag{3.5}$$

We have the following classification: we call hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma = [-2,0)$, very soft potentials if $\gamma \in (-3,-2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules and moderately soft potentials, i.e. $\gamma \in [-2,1]$.

The Landau equation conserves mass, momentum and energy. Indeed, at least formally, for any test function φ, we have

$$\int_{\mathbb{R}^3} Q(F,F)\varphi\, dv = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v-v_*) F F_* \left(\frac{\partial_i F}{F} - \frac{\partial_i F_*}{F_*} \right) (\partial_j \varphi - \partial_j \varphi_*)\, dv\, dv_*,$$

from which we deduce that for $\varphi(v) = 1, v, |v|^2$,

$$\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F \varphi(v)\, dx\, dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left[Q(F,F) - v \cdot \nabla_x F\right] \varphi(v)\, dx\, dv = 0. \tag{3.6}$$

Moreover, the Landau version of the Boltzmann H-theorem asserts that the entropy

$$H(F) := \int_{\mathbb{T}^3\times\mathbb{R}^3} F\,\log F\,dx\,dv$$

is non increasing. Indeed, at least formally, since a_{ij} is nonnegative, we have the following inequality for the entropy dissipation $D(F)$:

$$\begin{aligned} D(F) &:= -\frac{d}{dt}H(F) \\ &= \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*)FF_*\left(\frac{\partial_i F}{F}-\frac{\partial_i F_*}{F_*}\right)\left(\frac{\partial_j F}{F}-\frac{\partial_j F_*}{F_*}\right)\,dv\,dv_*\,dx \geq 0. \end{aligned}$$

It is known that the global equilibria of (3.1) are global Maxwellian distributions that are independent of time t and position x. We shall always consider initial data F_0 verifying

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F_0\,dx\,dv = 1, \quad \int_{\mathbb{T}^3\times\mathbb{R}^3} F_0\,v\,dx\,dv = 0, \quad \int_{\mathbb{T}^3\times\mathbb{R}^3} F_0\,|v|^2\,dx\,dv = 3,$$

therefore we consider the Maxwellian equilibrium

$$\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$$

with same mass, momentum and energy of the initial data.

We linearize the Landau equation around μ with the perturbation

$$F = \mu + f.$$

The Landau equation (3.1) for $f = f(t,x,v)$ takes the form

$$\begin{cases} \partial_t f = \Lambda f + Q(f,f) := \mathcal{L}f - v\cdot\nabla_x f + Q(f,f) \\ f_{|t=0} = f_0 = F_0 - \mu, \end{cases} \tag{3.7}$$

where $\Lambda = \mathcal{L} - v\cdot\nabla_x$ is the inhomogeneous linearized Landau operator and the homogeneous linearized Landau operator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}f &:= Q(\mu,f) + Q(f,\mu) \\ &= (a_{ij}*\mu)\;\partial_{ij}f - (c*\mu)f + (a_{ij}*f)\;\partial_{ij}\mu - (c*f)\mu. \end{aligned} \tag{3.8}$$

Through the paper we introduce the following notation

$$\bar a_{ij}(v) = a_{ij}*\mu, \quad \bar b_i(v) = b_i*\mu, \quad \bar c(v) = c*\mu. \tag{3.9}$$

The conservation laws (3.6) can then be rewritten as, for all $t\geq 0$,

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)\varphi(v)\,dx\,dv = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2. \tag{3.10}$$

3.1.2 Notations

Through all the paper we shall consider function of two variables $f = f(x, v)$ with $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$. Let $m = m(v)$ be a positive Borel weight function and $1 \le p, q \le \infty$. We define the space $L^q_x L^p_v(m)$ as the Lebesgue space associated to the norm, for $f = f(x, v)$,

$$\begin{aligned} \|f\|_{L^q_x L^p_v(m)} &:= \big\| \|f\|_{L^p_v(m)} \big\|_{L^q_x} := \big\| \|m\, f\|_{L^p_v} \big\|_{L^q_x} \\ &= \left(\int_{\mathbb{T}^3_v} \|f(x,\cdot)\|^q_{L^p_v(m)}\, dx \right)^{1/q} \\ &= \left(\int_{\mathbb{T}^3_v} \left(\int_{\mathbb{R}^3_x} |f(x,v)|^p\, m(v)^p\, dv \right)^{q/p} dx \right)^{1/q}. \end{aligned}$$

We also define the high-order Sobolev spaces $W^{n,q}_x W^{\ell,p}_v(m)$, for $n, \ell \in \mathbb{N}$:

$$\|f\|_{W^{n,q}_x W^{\ell,p}_v(m)} = \sum_{0 \le |\alpha| \le \ell,\, 0 \le |\beta| \le n,\, |\alpha|+|\beta| \le \max(\ell,n)} \|\partial^\alpha_v \partial^\beta_x f\|_{L^q_x L^p_v(m)}.$$

This definition reduces to the usual weighted Sobolev space $W^{\ell,p}_{x,v}(m)$ when $p = q$ and $\ell = n$, and we recall the shorthand notation $H^\ell = W^{\ell,2}$. In the case of negative Sobolev spaces we define the space $H^{-1}_{x,v}(m)$ associated to the norm

$$\|f\|_{H^{-1}_{x,v}(m)} := \|m\, f\|_{H^{-1}_{x,v}}$$

as well as $H^n_x H^{-1}_v(m)$, for $n \in \mathbb{N}$, associated with the norm

$$\|f\|_{H^n_x H^{-1}_v(m)} := \sum_{0 \le |\beta| \le n} \|\partial^\beta_x f\|_{L^2_x H^{-1}_v(m)} = \sum_{0 \le |\beta| \le n} \left(\int_{\mathbb{T}^3_x} \|m\, \partial^\beta_x f\|^2_{H^{-1}_v}\, dx \right)^{1/2}$$

We shall denote $W^{\ell,p}(m) = W^{\ell,p}_{x,v}(m)$ when considering spaces in the two variables (x, v). Moreover, we denote $W^{\ell,p}_x = W^{\ell,p}(\mathbb{T}^3_x)$ and its dual space is $W^{-\ell,p'}_x$ when considering only the x-variable. Similarly, $W^{\ell,p}_v(m) = W^{\ell,p}(\mathbb{R}^3_v; m)$ and its dual space is $W^{-\ell,p'}_v(m)$ when considering only the v-variable.

Let X, Y be Banach spaces and consider a linear operator $\Lambda : X \to X$. We shall denote by $\mathcal{S}_\Lambda(t) = e^{t\Lambda}$ the semigroup generated by Λ. Moreover we denote by $\mathscr{B}(X, Y)$ the space of bounded linear operators from X to Y and by $\|\cdot\|_{\mathscr{B}(X,Y)}$ its norm operator, with the usual simplification $\mathscr{B}(X) = \mathscr{B}(X, X)$.

For simplicity of notations, hereafter, we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$; $a \sim b$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 b \le a \le c_2 b$; we abbreviate " $\le C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed number.

3.1.3 Main and known results

Cauchy theory and convergence to equilibrium

We develop a Cauchy theory of perturbative solutions in "large" spaces for $\gamma \in [-2, 1]$. We also deal with the problem of convergence to equilibrium of the constructed solutions, we prove an exponential convergence to euqilibrium. Let us now state our assumptions for the main result.

(H0) Assumptions for Theorem 3.1.42:

– **Hard potentials** $\gamma \in (0, 1]$ **and Maxwellian molecules** $\gamma = 0$**:**

(i) *Polynomial weight*: $m = \langle v \rangle^k$ with $k > \gamma + 7 + 3/2$.

(ii) *Stretched exponential weight*: $m = e^{r\langle v \rangle^s}$ with $r > 0$ and $s \in (0, 2)$.

(iii) *Exponential weight*: $m = e^{r\langle v \rangle^2}$ with $r \in (0, 1/2)$.

– **Moderately soft potentials** $\gamma \in [-2, 0)$**:**

(i) *Stretched exponential wieght*: $m = e^{r\langle v \rangle^s}$ with $r > 0$, $s \in (-\gamma, 2)$.

(ii) *Exponential weight*: $m = e^{r\langle v \rangle^2}$ with $r \in (0, 1/2)$.

Through the paper, we shall use the notation $\sigma = 0$ when $m = \langle v \rangle^k$ and $\sigma = s$ when $m = e^{r\langle v \rangle^s}$.

We define the space $\mathcal{H}^3_x L^2_v(m)$ (for m a polynomial or exponential weight) associated to the norm

$$\begin{aligned} \|h\|^2_{\mathcal{H}^3_x L^2_v(m)} &= \|h\|^2_{L^2_x L^2_v(m)} + \|\nabla_x h\|^2_{L^2_x L^2_v(m\langle v \rangle^{-(1-\sigma/2)})} \\ &\quad + \|\nabla^2_x h\|^2_{L^2_x L^2_v(m\langle v \rangle^{-2(1-\sigma/2)})} + \|\nabla^3_x h\|^2_{L^2_x L^2_v(m\langle v \rangle^{-3(1-\sigma/2)})}. \end{aligned} \tag{3.11}$$

Moreover, we define in an similar way $\mathcal{H}^3_x H^1_{v,*}(m)$ associated to

$$\begin{aligned} \|h\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} &= \|h\|^2_{L^2_x H^1_{v,*}(m)} + \|\nabla_x h\|^2_{L^2_x H^1_{v,*}(m\langle v \rangle^{-(1-\sigma/2)})} \\ &\quad + \|\nabla^2_x h\|^2_{L^2_x H^1_{v,*}(m\langle v \rangle^{-2(1-\sigma/2)})} + \|\nabla^3_x h\|^2_{L^2_x H^1_{v,*}(m\langle v \rangle^{-3(1-\sigma/2)})}, \end{aligned} \tag{3.12}$$

where hereafter we introduce the notation

$$\|h\|^2_{H^1_{v,*}(m)} = \|h\|^2_{L^2_v(m\langle v \rangle^{(\gamma+\sigma)/2})} + \|P_v \nabla_v h\|^2_{L^2_v(m\langle v \rangle^{\gamma/2})} + \|(I - P_v)\nabla_v h\|^2_{L^2_v(m\langle v \rangle^{(\gamma+2)/2})}, \tag{3.13}$$

with P_v the projection onto v, namely $P_v \xi = (\xi \cdot \frac{v}{|v|}) \frac{v}{|v|}$.

Here are the main results on the fully nonlinear problem (3.7) that we prove in what follows. For simplicity denote $X := \mathcal{H}^3_x L^2_v(m)$ and $Y := \mathcal{H}^3_x H^1_{v,*}(m)$ (see (3.11) and (3.12)).

Theorem 3.1.42. *Consider assumption* **(H0)** *with some weight function m. We assume that f_0 satisfies* (3.10) *and also that $F_0 = \mu + f_0 \geq 0$. There is a constant $\epsilon_0 = \epsilon_0(m) > 0$ such that if $\|f_0\|_X \leq \epsilon_0$, then there exists a unique global weak solution f to the Landau equation* (3.7)*, which satisfies, for some constant $C > 0$,*

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \leq C\epsilon_0.$$

Moreover, this solution verifies an exponential decay: for any $0 < \lambda_2 < \lambda_1$ there exists $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f(t)\|_X \leq C\, e^{-\lambda_2 t}\, \|f_0\|_X,$$

where $\lambda_1 > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator in Theorem 3.2.45.

Let us comment our result and give an overview on the previous works on the Cauchy theory for the inhomogeneous Landau equation. For general large data, we refer to the papers of DiPerna-Lions [39] for global existence of the so-called renormalized solutions in the case of the Boltzmann equation. This notion of solution have been extend to the Landau equation by Alexandre-Villani [3] where they construct global renormalized solutions with a defect measure. We also mention the work of Desvillettes-Villani [38] that proves the convergence to equilibrium of a priori smooth solutions for both Boltzmann and Landau equations for general initial data.

In a close-to-equilibrium framework, Guo in [52] has developed a theory of perturbative solutions in a space with a weight prescribed by the equilibrium of type $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 8$, and for all cases $\gamma \in [-3, 1]$, using an energy method. Later, for $\gamma \in [-2, 1]$, Mouhot-Neumann [76] improve this result to $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 4$.

Let us underline the fact that Theorem 3.1.42 largely improves previous results on the Cauchy theory associated to the Landau equation in a perturbative setting. Indeed, we considerably have enlarged the space in which the Cauchy theory has been developed in two ways: the weight of our space is much less restrictive (it can be a polynomial or stretched exponential weight instead of the inverse Maxwellian equilibrium) and we also require less assumptions on the derivatives, in particular no derivatives in the velocity variable.

Moreover, we also deal with the problem of the decay to equilibrium of the solutions that we construct. This problem has been considered in several papers by Guo and Strain in [84, 85] first for Coulombian interactions ($\gamma = -3$) for which they proved an almost exponential decay and then, they have improved this result dealing with very soft potentials ($\gamma \in [-3, -2)$) and proving a decay to equilibrium with a rate of type $e^{-\lambda t^p}$ with $p \in (0, 1)$. In the case $\gamma \in [-2, 1]$, Yu [103] has proved an exponential decay in $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 8$, and Mouhot-Neumann [76] in $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 4$.

We here emphasize that our strategy to prove Theorem 3.1.42 is completely different from the one of Guo in [52]. Indeed, he uses an energy method and his strategy is purely nonlinear, he directly derives energy estimates for the nonlinear problem while the first step of our proof is the study of the linearized equation and more precisely the study of its spectral properties. Then, we go back to the nonlinear problem combining the new spectral estimates obtained on the linearized equation with some bilinear estimates on the collision operator. Thanks to this method, we are able to develop a Cauchy theory in a space which is much larger than the one from the previous paper [52]. Moreover, we obtain the convergence of solutions towards the equilibrium with an explicit exponential rate.

Since the study of the linearized equation is the cornerstone of the proof of our main result, we here present the result that we obtain on it and briefly remind previous results.

The linearized equation.

We remind the definition of the linearized operator at first order around the equilibrium:

$$\Lambda f = Q(\mu, f) + Q(f, \mu) - v \cdot \nabla_x f.$$

We study spectral properties of the linearized operator Λ in various weighted Sobolev spaces $W^{n,p}_x W^{\ell,p}_v$. Let us state our main result on the linearized operator (see Theorem 3.2.45 for a precise statement), which widely generalizes previous results since we are able to deal with a more general class of spaces.

Theorem 3.1.43. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *defined in Subsection 3.2.1 and a weight function m. Let $\mathcal{E}$ be one of the admissible spaces defined in* (3.19). *Then, there exists explicit constants $\lambda_1 > 0$ and $C > 0$ such that*

$$\forall\, t \geq 0, \quad \forall\, f \in \mathcal{E}, \quad \|S_\Lambda(t) f - \Pi_0 f\|_{\mathcal{E}} \leq C\, e^{-\lambda_1 t}\, \|f - \Pi_0 f\|_{\mathcal{E}},$$

where $S_\Lambda(t)$ is the semigroup associated to Λ and Π_0 the projector onto the null space of Λ by (3.16).

We first make a brief review on known results on spectral gap properties of the linearized operator in the homogeneous $\mathcal{L}$ defined in (3.8). On the Hilbert space $L^2_v(\mu^{-1/2})$, a simple computation gives that $\mathcal{L}$ is self-adjoint and $\langle \mathcal{L} h, h\rangle_{L^2_v(\mu^{-1/2})} \leq 0$, which implies that the spectrum of $\mathcal{L}$ on $L^2_v(\mu^{-1})$ is included in $\mathbb{R}^-$. Moreover, the nullspace is given by

$$N(\mathcal{L}) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}.$$

We can now state the existing results on the spectral gap of $\mathcal{L}$ on $L^2_v(\mu^{-1/2})$. Summarising results of Degond and Lemou [34], Guo [52], Baranger and Mouhot [15], Mouhot [74], Mouhot and Strain [77] for all cases $\gamma \in [-3, 1]$, we have: there is a constructive constant $\lambda_0 > 0$ (spectral gap) such that

$$\langle -\mathcal{L} h, h\rangle_{L^2_v(\mu^{-1/2})} \geq \lambda_0 \|h\|^2_{H^1_{v,**}(\mu^{-1/2})}, \quad \forall\, h \in N(\mathcal{L})^\perp, \tag{3.14}$$

where the anisotropic norm $\|\cdot\|_{H^1_{v,**}(\mu^{-1/2})}$ is defined by

$$\begin{aligned}\|h\|^2_{H^1_{v,**}(\mu^{-1/2})} := \|\langle v\rangle^{\gamma/2} P_v \nabla h\|^2_{L^2_v(\mu^{-1/2})} + \|\langle v\rangle^{(\gamma+2)/2} (I - P_v)\nabla h\|^2_{L^2_v(\mu^{-1/2})} \\ + \|\langle v\rangle^{(\gamma+2)/2} h\|^2_{L^2_v(\mu^{-1/2})},\end{aligned}$$

where P_v denotes the projection onto the v-direction, more precisely $P_v g = \left(\frac{v}{|v|} \cdot g\right) \frac{v}{|v|}$. We also have from [52] the reverse inequality, which implies a spectral gap for $\mathcal{L}$ in $L^2_v(\mu^{-1/2})$ if and only if $\gamma + 2 \geq 0$.

Let us now mention the works which have studied spectral properties of the full linearized operator $\Lambda = \mathcal{L} - v \cdot \nabla_x$. Mouhot and Neumann [76] prove explicit coercivity estimates for hard and moderately soft potentials ($\gamma \in [-2, 1]$) in $H^\ell_{x,v}(\mu^{-1/2})$ for $\ell \geq 1$, using the known spectral estimate for $\mathcal{L}$ in (3.14). It is worth mentioning that the third author has obtained in [101] an exponential decay to equilibrium for the full linearized equation in $L^2_{x,v}(\mu^{-1/2})$ by a different method, and the decay rate depends on the size of the domain. Let us summarize results that we will use in the remainder of the paper in the following theorem.

Theorem 3.1.44 ([76]). *Consider $\ell_0 \geq 1$ and $E := H^{\ell_0}_{x,v}(\mu^{-1/2})$. Then, there exists a constructive constant $\lambda_0 > 0$ (spectral gap) such that Λ satisfies on E:*

(i) the spectrum $\Sigma(\Lambda) \subset \{z \in \mathbb{C} : \Re e z \leq -\lambda_0\} \cup \{0\}$;

(ii) the null space $N(\Lambda)$ is given by

$$N(\Lambda) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}, \tag{3.15}$$

and the projection Π_0 onto $N(\Lambda)$ by

$$\begin{aligned}\Pi_0 f = &\left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} f\, dx\, dv\right)\mu + \sum_{i=1}^3 \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} v_i f\, dx\, dv\right) v_i \mu \\ &+ \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|v|^2 - 3}{18} f\, dx\, dv\right) \frac{(|v|^2 - 3)}{18}\, \mu;\end{aligned} \tag{3.16}$$

(iii) Λ is the generator of a strongly continuous semigroup $S_\Lambda(t)$ that satisfies

$$\forall\, t \geq 0,\ \forall\, f \in E, \quad \|S_\Lambda(t) f - \Pi_0 f\|_E \leq e^{-\lambda_0 t} \|f - \Pi_0 f\|_E. \tag{3.17}$$

To prove Theorem 3.1.43, our strategy follows the one initiated by Mouhot in [75] for the homogeneous Boltzmann equation for hard potentials with cut-off. The latter theory has then been developed and extend in an abstract setting by Gualdani, Mischler and Mouhot [51], and Mischler and Mouhot [64]. They have applied it to Fokker-Planck equations and the spatially inhomogeneous Boltzmann equation for hard spheres. This strategy has also been used for the homogeneous Landau equation for hard and moderately soft potentials by the first author in [25, 26] and by the second author for the fractional Fokker-Planck equation and the homogeneous Boltzmann equation for hard potentials without cut-off in [90, 89] (see also [70] for related works).

Let us describe in more details this strategy. We want to apply the abstract theorem of enlargement of the space of semigorup decay from [51, 64] to our linearized operator Λ. We shall deduce the spectral/semigroup estimates of Theorem 3.1.43 on "large spaces" $\mathcal{E}$ using the already known spectral gap estimates for Λ on $H^\ell_{x,v}(\mu^{-1/2})$, for $\ell \geq 1$, described in Theorem 3.1.44. Roughly speaking, to do that, we have to find a splitting of Λ into two operators $\Lambda = \mathcal{A} + \mathcal{B}$ which satisfy some properties. The first part $\mathcal{A}$ has to be bounded, the second one $\mathcal{B}$ has to have some dissipativity properties, and also the semigroup $(\mathcal{A}S_{\mathcal{B}}(t))$ is required to have some regularization properties.

We end this introduction by describing the organization of the paper. In Section 3.2 we consider the linearized equation and prove a precise version of Theorem 3.1.43. In Section 3.3 we come back to the nonlinear equation and prove our main result Theorem 3.1.42.

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3.2 The linearized equation

3.2.1 Functional spaces and main results on the linearized equation

Let us now make our assumptions on the different potentials γ and weight functions $m = m(v)$:

(H1) Hard potentials $\gamma \in (0,1]$. For $p \in [1,\infty]$ we consider the following cases

(i) *Polynomial weight*: let $m = \langle v\rangle^k$ with $k > \gamma + 2 + 3(1 - 1/p)$, and define the abscissa $\lambda_{m,p} := \infty$.

(ii) *Stretched exponential weight*: let $m = e^{r\langle v\rangle^s}$ with $r > 0$ and $s \in (0,2)$, and define the abscissa $\lambda_{m,p} := \infty$.

(iii) *Exponential weight*: let $m = e^{r\langle v\rangle^2}$ with $r \in (0,1/2)$ and define the abscissa $\lambda_{m,p} := \infty$.

(H2) Maxwellian molecules $\gamma = 0$. For $p \in [1,\infty]$ we consider the following cases

(i) *Polynomial weight*: let $m = \langle v\rangle^k$ with $k > \gamma + 2 + 3(1 - 1/p)$ and $k > \gamma + 4 + 3/2$ and define the abscissa $\lambda_{m,p} := 2[k - (\gamma + 3)(1 - 1/p)]$.

(ii) *Stretched exponential weight*: let $m = e^{r\langle v\rangle^s}$ with $r > 0$ and $s \in (0,2)$, and define the abscissa $\lambda_{m,p} := \infty$.

(iii) *Exponential weight*: let $m = e^{r\langle v\rangle^2}$ with $r \in (0,1/2)$ and define the abscissa $\lambda_{m,p} := \infty$.

(H3) Moderately soft potentials $\gamma \in [-2,0)$. For $p \in [1,\infty]$ we consider the following cases

(i) *Stretched exponential weight for* $\gamma \in (-2,0)$: let $m = e^{r\langle v\rangle^s}$ with $r > 0$, $s \in (0,2)$ and $s + \gamma > 0$, and define the abscissa $\lambda_{m,p} := \infty$.

(ii) *Exponential weight for* $\gamma \in (-2,0)$: let $m = e^{r\langle v\rangle^2}$ with $r \in (0,1/2)$ and define the abscissa $\lambda_{m,p} := \infty$.

(iii) *Exponential weight for* $\gamma = -2$: let $m = e^{r\langle v\rangle^2}$ with $r \in (0,1/2)$, and define the abscissa $\lambda_{m,p} := 4r(1-2r)$.

Under these hypothesis, we shall use the following notation for the functional spaces:

$$E := H^{\ell_0}_{x,v}(\mu^{-1/2}), \quad \ell_0 \geq 1, \tag{3.18}$$

in which space we already know that the linearized operator Λ has a spectral gap (Theorem 3.1.44), and also, under hypotheses **(H1)**, **(H2)** or **(H3)**,

$$\mathcal{E} := \begin{cases} L^p_{x,v}(m), & \forall\, p \in [1,\infty]; \\ W^{n,p}_x W^{\ell,p}_v(m), & \forall\, p \in [1,2],\, n \in \mathbb{N}^*,\, \ell \in \mathbb{N}; \\ H^n_x H^{-1}_v(m), & \forall\, n \in \{-1\} \cup \mathbb{N}; \end{cases} \tag{3.19}$$

and for each space we define the associated abscissa $\lambda_{\mathcal{E}} = \lambda_{m,p}$.

The main result of this section, which is a precise version of Theorem 3.1.43, reads

Theorem 3.2.45. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *with some weight m, and let $\mathcal{E}$ be one of the admissible spaces defined in* (3.19).

Then, for any $\lambda < \lambda_{\mathcal{E}}$ and any $\lambda_1 \leq \min\{\lambda_0, \lambda\}$, where we recall that $\lambda_0 > 0$ is the spectral gap of Λ on E (see (3.17)*), there is a constructive constant $C > 0$ such that the operator Λ satisfies on $\mathcal{E}$:*

(i) $\Sigma(\Lambda) \subset \{z \in \mathbb{C} \mid \Re e\, z \leq -\lambda_1\} \cup \{0\}$;

(ii) the null-space $N(\Lambda)$ is given by (3.15) *and the projection Π_0 onto $N(\Lambda)$ by* (3.16);

(iii) Λ is the generator of a strongly continuous semigroup $S_\Lambda(t)$ that verifies

$$\forall\, t \geq 0, \forall\, f \in \mathcal{E}, \quad \|S_\Lambda(t) f - \Pi_0 f\|_{\mathcal{E}} \leq C\, e^{-\lambda_1 t}\, \|f - \Pi_0 f\|_{\mathcal{E}}.$$

Remark 3.2.46. *(1) Observe that:*

- *Cases* **(H1)**, **(H2)-(ii)-(iii)** *or* **(H3)-(i)-(ii)**: *we can recover the optimal estimate $\lambda_1 = \lambda_0$ since $\lambda_{m,p} = +\infty$.*
- *Case* **(H2)-(i)**: *in this case we have $m = \langle v\rangle^k$, and we can recover the optimal estimate $\lambda_1 = \lambda_0$ if $k > 0$ is large enough such that $\lambda_{m,p} = 2k - 6(1-1/p) > \lambda_0$. Otherwise, we obtain $\lambda_1 < 2k - 6(1-1/p)$.*
- *Case* **(H3)-(iii)**: *in this case we have $\gamma = -2$, $m = e^{r\langle v\rangle^2}$ and $\lambda_{m,p} = 4r(1-2r)$ and the condition $0 < r < 1/2$.*

(2) This theorem also holds for other choices of space, namely for a space $\mathcal{E}$ that is an interpolation space of two admissible spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ in (3.19)*. We will use this on Section 3.3 to study the nonlinear equation.*

The proof of Theorem 3.2.45 uses the fact that the properties (i)-(ii)-(iii) with $\lambda_1 = \lambda_0$ hold on the small space E (Theorem 3.1.44) and the strategy described in Section 3.1.3.

3.2.2 Splitting of the linearized operator

We decompose the homogeneous linearized Landau operator $\mathcal{L}$ defined in (3.8) as $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$, where we define

$$\mathcal{A}_0 f := (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu, \qquad \mathcal{B}_0 f := (a_{ij} * \mu)\partial_{ij} f - (c * \mu) f. \tag{3.20}$$

Consider a smooth positive function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^3_v)$ such that $0 \le \chi(v) \le 1$, $\chi(v) \equiv 1$ for $|v| \le 1$ and $\chi(v) \equiv 0$ for $|v| > 2$. For any $R \ge 1$ we define $\chi_R(v) := \chi(R^{-1}v)$ and in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$.

Then, we make the final decomposition of the operator Λ as $\Lambda = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A} := \mathcal{A}_0 + M\chi_R, \qquad \mathcal{B} := \mathcal{B}_0 - v \cdot \nabla_x - M\chi_R, \tag{3.21}$$

where $M > 0$ and $R > 0$ will be chosen later (see Lemma 3.2.50).

3.2.3 Preliminaries

We have the following results concerning the matrix $\bar{a}_{ij}(v)$.

Lemma 3.2.47. *The following properties hold:*

(a) *The matrix $\bar{a}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace $v^\perp$. Moreover, when $|v| \to +\infty$ we have*

$$\ell_1(v) \sim 2\langle v\rangle^\gamma \quad \text{and} \quad \ell_2(v) \sim \langle v\rangle^{\gamma+2}.$$

(b) *The function $\bar{a}_{ij}$ is smooth, for any multi-index $\beta \in \mathbb{N}^3$*

$$|\partial^\beta \bar{a}_{ij}(v)| \le C_\beta \langle v\rangle^{\gamma+2-|\beta|}$$

and

$$\begin{aligned}\bar{a}_{ij}(v)\xi_i\xi_j &= \ell_1(v)|P_v\xi|^2 + \ell_2(v)|(I - P_v)\xi|^2 \\ &\ge c_0\{\langle v\rangle^\gamma |P_v\xi|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\xi|^2\},\end{aligned}$$

for some constant $c_0 > 0$ and where P_v is the projection on v: $P_v\xi_i = \left(\xi \cdot \frac{v}{|v|}\right)\frac{v_i}{|v|}$.

(c) *We have*

$$\bar{a}_{ii}(v) = \mathrm{tr}(\bar{a}(v)) = \ell_1(v) + 2\ell_2(v) = 2\int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\mu(v_*)\, dv_*$$

and

$$\bar{b}_i(v) = -\ell_1(v)\, v_i.$$

(d) *If $|v| > 1$, we have*

$$|\partial^\beta \ell_1(v)| \le C_\beta \langle v\rangle^{\gamma-|\beta|} \qquad \text{and} \qquad |\partial^\beta \ell_2(v)| \le C_\beta \langle v\rangle^{\gamma+2-|\beta|}.$$

Proof. We just give the proof of item (d) since (a) comes from [34, Propositions 2.3 and 2.4, Corollary 2.5], (b) is [52, Lemma 3] and (c) is evident. For item (d), the estimate of $|\partial^\beta \ell_2(v)|$ directly comes from (a) and [52, Lemma 2]. For $\ell_1(v)$, using (b) and (c),

$$\partial_v \bar{b}_i(v) = \partial_v\big(- \ell_1(v) v_i\big)\,,$$

and hence

$$|\partial_v \ell_1(v)||v| \le C\big(|\ell_1(v)| + |\partial_v \bar{b}_i(v)|\big) \le C\langle v\rangle^\gamma\,,$$

note that $|v| > 1$, we thus have

$$|\partial_v \ell_1(v)| \le C|v|^{-1}\langle v\rangle^\gamma \le C\langle v\rangle^{\gamma-1}\,.$$

The high order estimate is similar and hence we omit the details. □

The following elementary lemma will be useful in the sequel (see [25, Lemma 2.5] and [26, Lemma 2.3]).

Lemma 3.2.48. *Let $J_\alpha(v) := \int_{\mathbb{R}^3} |v-w|^\alpha \mu(w)\, dw$, for $-3 \le \alpha \le 3$. Then it holds:*

(a) If $2 < \alpha \le 3$, then $J_\alpha(v) \le |v|^\alpha + C_\alpha |v|^{\alpha/2} + C_\alpha$, for some constant $C_\alpha > 0$.

(b) If $0 \le \alpha \le 2$, then $J_\alpha(v) \le |v|^\alpha + C_\alpha$, for some constant $C_\alpha > 0$.

(c) If $-3 < \alpha < 0$, then $J_\alpha(v) \le C\langle v\rangle^\alpha$ for some constant $C > 0$.

We define the function $\varphi_{m,p}$ as

$$\varphi_{m,p}(v) := \bar{a}_{ij}(v)\frac{\partial_{ij} m}{m} + (p-1)\bar{a}_{ij}(v)\frac{\partial_i m}{m}\frac{\partial_j m}{m} + 2\bar{b}_i(v)\frac{\partial_i m}{m} + \left(\frac{1}{p} - 1\right)\bar{c}(v), \quad (3.22)$$

and also the function $\tilde{\varphi}_{m,p}$ given by

$$\begin{aligned}\tilde{\varphi}_{m,p}(v) := &\left(\frac{2}{p} - 1\right)\bar{a}_{ij}(v)\frac{\partial_{ij} m}{m} + \left(2 - \frac{2}{p}\right)\bar{a}_{ij}(v)\frac{\partial_i m}{m}\frac{\partial_j m}{m}\\ &+ \frac{2}{p}\,\bar{b}_i(v)\frac{\partial_i m}{m} + \left(\frac{1}{p} - 1\right)\bar{c}(v),\end{aligned} \quad (3.23)$$

and hereafter, in order to treat both weight functions at the same time, we recall the notation: $\sigma = 0$ when $m = \langle v\rangle^k$ and $\sigma = s$ when $m = e^{r\langle v\rangle^s}$.

We prove the following result concerning $\varphi_{m,p}$ and $\tilde{\varphi}_{m,p}$.

Lemma 3.2.49. *Consider* **(H1)**, **(H2)** *or* **(H3)**, *and let $\varphi_{m,p}$ and $\tilde{\varphi}_{m,p}$ be defined in* (3.22) *and* (3.23) *respectively. Then we have:*

- *Assume $\sigma \in [0, 2)$:*

(1) For all positive $\lambda < \lambda_{m,p}$ and $\delta \in (0, \lambda_{m,p} - \lambda)$ we can choose M and R large enough such that

$$\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

$$\tilde{\varphi}_{m,p}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

(2) For all positive $\lambda < \lambda_{m,p}$ and $\delta \in (0, \lambda_{m,p} - \lambda)$ we can choose M and R large enough such that

$$\varphi_{m,p}(v) - M\chi_R(v) + M\partial_j\chi_R(v) \leq -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

$$\tilde\varphi_{m,p}(v) - M\chi_R(v) + M\partial_j\chi_R(v) \leq -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

• *Assume $\sigma = 2$: The same conclusion as before holds for $\tilde\varphi_{m,p}$. Moreover, concerning $\varphi_{m,p}$, the previous estimates also hold if we restrict $r \in (0, 1/(2p))$ in assumptions* **(H1)-(iii), (H2)-(iii), (H3)-(ii)**, *and also modifying the value of the abscissa $\lambda_{m,p} = 4r(1-2rp)$ in* **(H3)-(iii)**.

Proof of Lemma 3.2.49. Step 1. Polynomial weight. Consider $m = \langle v\rangle^k$ under hypothesis **(H1)** or **(H2)**. On the one hand, we have

$$\frac{\partial_i m}{m} = kv_i\langle v\rangle^{-2}, \qquad \frac{\partial_i m}{m}\frac{\partial_j m}{m} = k^2 v_i v_j\langle v\rangle^{-4},$$
$$\frac{\partial_{ij} m}{m} = \delta_{ij}\, k\langle v\rangle^{-2} + k(k-2)v_i v_j\langle v\rangle^{-4}.$$

Hence, from definitions (3.4)-(3.9) and Lemma 3.2.47 we obtain

$$\begin{aligned}\bar a_{ij}\,\frac{\partial_{ij} m}{m} &= (\delta_{ij}\bar a_{ij})\, k\langle v\rangle^{-2} + (\bar a_{ij}v_iv_j)\, k(k-2)\langle v\rangle^{-4}\\ &= \bar a_{ii}\, k\langle v\rangle^{-2} + \ell_1(v)\, k(k-2)|v|^2\langle v\rangle^{-4},\end{aligned}$$

where we recall that the eigenvalue $\ell_1(v) > 0$ is defined in Lemma 3.2.47. Moreover, arguing exactly as above we obtain

$$\bar a_{ij}\,\frac{\partial_i m}{m}\frac{\partial_j m}{m} = (\bar a_{ij}v_iv_j)\, k^2\langle v\rangle^{-4} = \ell_1(v)\, k^2|v|^2\langle v\rangle^{-4}$$

and also, using the fact that $\bar b_i(v) = -\ell_1(v)v_i$ from Lemma 3.2.47,

$$\bar b_i\,\frac{\partial_i m}{m} = -\ell_1(v)v_i\, kv_i\langle v\rangle^{-2} = -\ell_1(v)\, k|v|^2\langle v\rangle^{-2}.$$

On the other hand, from item (c) of Lemma 3.2.47 and definitions (3.4)-(3.9) we obtain that

$$\bar a_{ii}(v) = \ell_1(v) + 2\ell_2(v) \quad\text{and}\quad \bar c(v) = -2(\gamma+3)J_\gamma(v),$$

where J_α is defined in Lemma 3.2.48. It follows that

$$\begin{aligned}\varphi_{m,p}(v) &= 2k\ell_2(v)(v)\langle v\rangle^{-2} + k\ell_1(v)\langle v\rangle^{-2} + k(k-2)\,\ell_1(v)\,|v|^2\langle v\rangle^{-4}\\ &\quad + (p-1)k^2\,\ell_1(v)\,|v|^2\langle v\rangle^{-4} - 2k\,\ell_1(v)\,|v|^2\langle v\rangle^{-2} + 2(\gamma+3)\left(1-\frac{1}{p}\right)J_\gamma(v).\end{aligned} \tag{3.24}$$

Since $\ell_1(v) \sim 2\langle v\rangle^\gamma$, $\ell_2(v) \sim \langle v\rangle^{\gamma+2}$ and $\ell_1(v)|v|^2 \sim 2\ell_2(v)$ when $|v| \to +\infty$ thanks to Lemma 3.2.47, and also $J_\gamma(v) \sim \langle v\rangle^\gamma$ from Lemma 3.2.48 (since in this case we have $\gamma \geq 0$), the dominant terms in (3.24) are the first, fifth and sixth ones, all of order $\langle v \rangle^\gamma$. Then we obtain

$$\limsup_{|v|\to+\infty} \varphi_{m,p}(v) \leq -2\left[k - (\gamma+3)(1-1/p)\right]\langle v \rangle^\gamma, \tag{3.25}$$

and recall that $k > (\gamma+3)(1-1/p)$. Doing the same kind of computations, we obtain the same asymptotic for $\tilde{\varphi}_{m,p}$,

$$\limsup_{|v|\to+\infty} \tilde{\varphi}_{m,p}(v) \leq -2[k - (\gamma+3)(1-1/p)]\langle v \rangle^\gamma. \tag{3.26}$$

Step 2. Stretched exponential weight. We consider now $m = \exp(r\langle v \rangle^s)$ satisfying **(H1)**, **(H2)** or **(H3)**. In this case we have

$$\frac{\partial_i m}{m} = rsv_i\langle v \rangle^{s-2}, \quad \frac{\partial_i m}{m}\frac{\partial_j m}{m} = r^2s^2v_iv_j\langle v \rangle^{2s-4},$$
$$\frac{\partial_{ij} m}{m} = rs\langle v \rangle^{s-2}\delta_{ij} + rs(s-2)v_iv_j\langle v \rangle^{s-4} + r^2s^2v_iv_j\langle v \rangle^{2s-4}.$$

Then we obtain

$$\begin{aligned}\varphi_{m,p}(v) = {} & 2rs\,\ell_2(v)\langle v \rangle^{s-2} + rs\,\ell_1(v)\langle v \rangle^{s-2} + rs(s-2)\,\ell_1(v)|v|^2\langle v \rangle^{s-4} \\ & + pr^2s^2\,\ell_1(v)|v|^2\langle v \rangle^{2s-4} - 2rs\,\ell_1(v)|v|^2\langle v \rangle^{s-2} + 2(\gamma+3)\left(1-\frac{1}{p}\right)J_\gamma(v).\end{aligned} \tag{3.27}$$

In the case $0 < s < 2$, arguing as in step 1, the dominant terms in (3.27) when $|v| \to +\infty$ are the first and fifth one, both of order $\langle v \rangle^{\gamma+s}$. Then we obtain

$$\limsup_{|v|\to+\infty} \varphi_{m,p}(v) \leq -2rs\langle v \rangle^{\gamma+s}, \tag{3.28}$$

and recall that $s+\gamma > 0$. In the same way we obtain

$$\limsup_{|v|\to+\infty} \tilde{\varphi}_{m,p}(v) \leq -2rs\langle v \rangle^{\gamma+s}. \tag{3.29}$$

In the case $s = 2$, the dominant terms in (3.27) when $|v| \to +\infty$ are the first, fourth and fifth ones, all of order $\langle v \rangle^{\gamma+2}$. Hence we get

$$\limsup_{|v|\to+\infty} \varphi_{m,p}(v) \leq -4r(1-2pr)\langle v \rangle^{\gamma+2}. \tag{3.30}$$

However, a similar computation gives

$$\limsup_{|v|\to+\infty} \tilde{\varphi}_{m,p}(v) \leq -4r(1-2r)\langle v \rangle^{\gamma+2}, \tag{3.31}$$

which is better than the asymptotic of $\varphi_{m,p}$. Thus we need the condition $r < 1/2$ for $\tilde{\varphi}_{m,p}$ (which is better than the condition $r < 1/(2p)$ for $\varphi_{m,p}$).

Step 3. Conclusion. Finally, thanks to the asymptotic behaviour in (3.25), (3.28) and (3.30), for any $\lambda < \lambda_{m,p}$ we can choose M and R large enough such that

$$\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}$$

for some $\delta > 0$ small enough, which gives us point (1) of the lemma.

For the point (2) we use $\partial_j\chi_R(v) = R^{-1}\partial_j\chi(v/R)$ and write

$$\varphi_{m,p}(v) - M\chi_R(v) + M\partial_j\chi_R(v) \le \varphi_{m,p}(v) - M\chi_R(v) + M\frac{C_\chi}{R}\,\mathbf{1}_{R\le|v|\le 2R} =: \Phi(v).$$

We fix some $\bar\lambda \in (\lambda, \lambda_{m,p})$. First we choose R_1 large enough such that, for all $|v| \ge R_1$, we have

$$\varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} \le -\bar\lambda$$

for some $\delta > 0$ small enough, which implies that, for any $|v| \ge 2R_1$,

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} = \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} \le -\bar\lambda.$$

Then we choose $M > 0$ large enough such that, for all $|v| \le R_1$,

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} = \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} - M\chi_{R_1}(v) \le -\bar\lambda.$$

Finally, we choose $R \ge R_1$ large enough such that, for any $R \le |v| \le 2R$,

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} \le \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} + M\frac{C_\chi}{R} \le -\bar\lambda + M\frac{C_\chi}{R} \le -\lambda,$$

and we easily observe that now for $R_1 \le |v| \le R$ we have

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} = \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} - M\chi_R(v) \le -\bar\lambda - M \le -\lambda,$$

which concludes the proof for $\varphi_{m,p}$. Concerning $\tilde\varphi_{m,p}$, in the same way, inequalities (3.26), (3.29) and (3.31) yield the result. □

3.2.4 Hypodissipativity

In this subsection we prove hypodissipativity properties for the operator $\mathcal{B}$ on the admissible spaces $\mathcal{E}$ defined in (3.19).

Lemma 3.2.50. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *and let $p \in [1, +\infty]$. Then, for any $\lambda < \lambda_{m,p}$, we can choose $M > 0$ and $R > 0$ large enough such that the operator $(\mathcal{B} + \lambda)$ is dissipative in $L^p_{x,v}(m)$, in the sense that*

$$\forall\, t \ge 0, \quad \|S_{\mathcal{B}}(t)\|_{\mathscr{B}(L^p_{x,v}(m))} \le Ce^{-\lambda t}.$$

Proof of Lemma 3.2.50. Let us denote $\Phi'(z) = |z|^{p-1}\mathrm{sign}(z)$ and consider the equation

$$\partial_t f = \mathcal{B} f = \mathcal{B}_0 f - v \cdot \nabla_x f - M\chi_R f.$$

For all $p \in [1, +\infty)$, we have

$$\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p_{x,v}(m)} = \int (\mathcal{B} f)\Phi'(f)\, m^p.$$

From (3.5) and (3.20), last integral is equal to

$$\begin{aligned}&\int \bar a_{ij}(v)\partial_{ij} f(x,v)\Phi'(f) m^p - \int \bar c(v) f(x,v)\Phi'(f) m^p \\ &\quad - \int v\cdot\nabla_x f(x,v)\Phi'(f)m^p - \int M\chi_R(v) f(x,v)\Phi'(f) m^p \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

The term T_3 vanishes thanks to its divergence structure and terms T_2 and T_4 are easily computed, giving

$$T_2 = -\int \bar c(v)|f(x,v)|^p m^p \quad \text{and} \quad T_4 = -\int M\chi_R(v)|f(x,v)|^p m^p.$$

Let us compute then the term T_1. Using that $\partial_{ij} f \Phi'(f) = p^{-1}\partial_{ij}(|f|^p) - (p-1)\partial_i f \partial_j f |f|^{p-2}$ we obtain

$$T_1 = \frac{1}{p}\int \bar a_{ij}(v)\partial_{ij}(|f|^p) m^p - (p-1)\int \bar a_{ij}(v)\partial_i f \partial_j f\, |f|^{p-2} m^p.$$

Performing two integrations by parts on the first integral of T_1 it yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p_{x,v}(m)} = \int (\mathcal{B} f)\Phi'(f)\, m^p &= -(p-1)\int \bar a_{ij}(v)\partial_i f\partial_j f\, |f|^{p-2} m^p \\ &\quad + \int \{\varphi_{m,p}(v) - M\chi_R(v)\}\, |f|^p\, m^p,\end{aligned}$$

where $\varphi_{m,p}$ is defined in (3.22). We can also get, by a similar computation,

$$\begin{aligned}\int (\mathcal{B} f)\Phi'(f)\, m^p = -(p-1)&\int \bar a_{ij}(v)\partial_i(mf)\partial_j(mf)\, |f|^{p-2} m^{p-2} \\ &+ \int \{\tilde\varphi_{m,p}(v) - M\chi_R(v)\}\, |f|^p\, m^p.\end{aligned}$$

Finally, thanks to Lemma 3.2.49, for any $\lambda < \lambda_{m,p}$ we can choose M and R large enough such that $\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda + \delta\langle v\rangle^{\gamma+\sigma}$. It follows that the operator $\mathcal{B} + \lambda$ is dissipative in $L^p_{x,v}(m)$, more precisely, for all $f \in L^p_{x,v}(m)$, we have

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^p_{x,v}(m)} \le e^{-\lambda t}\|f\|_{L^p_{x,v}(m)}. \tag{3.32}$$

Indeed we obtain

$$\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p(m)} \le -c_0(p-1)\int \{\langle v\rangle^\gamma |P_v\nabla_v f|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v f|^2\}\,|f|^{p-2}\,m^p - \lambda\|f\|^p_{L^p(m)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{p}} f\|^p_{L^p(m)}. \tag{3.33}$$

or

$$\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p(m)} \le -c_0(p-1)\int \{\langle v\rangle^\gamma |P_v\nabla_v (mf)|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v (mf)|^2\}\,|f|^{p-2}\,m^{p-2} - \lambda\|f\|^p_{L^p(m)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{p}} f\|^p_{L^p(m)}, \tag{3.34}$$

from which (3.32) follows for any $p \in [1,\infty)$. For $p=\infty$, let $g = mf$, it is easy to check that g satisfies the equation

$$\partial_t g + v\cdot\nabla_x g = \bar a_{ij}(v)\partial_{ij} g - 2\bar a_{ij}(v)\frac{\partial_i m}{m}\partial_j g + \tilde\varphi_{m,\infty}(v) g - M\chi_R(v) g\,,$$

by the standard maximum principle argument (for example, see [102]), we have

$$\|\mathcal{S}_\mathcal{B}(t) f\|_{L^\infty_{x,v}(m)} \le e^{-\lambda t}\|f\|_{L^\infty_{x,v}(m)}.$$

This completes the proof of the lemma. □

Lemma 3.2.51. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, $\ell \in \mathbb{N}$ *and* $n \in \mathbb{N}^*$. *Then, for any* $\lambda < \lambda_{m,1}$, *we can choose* $M>0$ *and* $R>0$ *large enough such that the operator* $\mathcal{B}+\lambda$ *is hypo-dissipative in* $W^{n,1}_x W^{\ell,1}_v(m)$, *in the sense that*

$$\forall\, t\ge 0,\quad \|S_\mathcal{B}(t)\|_{\mathscr{B}(W^{n,1}_x W^{\ell,1}_v(m))} \le C e^{-\lambda t}.$$

Proof of Lemma 3.2.51. Consider the equation

$$\partial_t f = \mathcal{B} f = \mathcal{B}_0 f - v\cdot\nabla_x f - M\chi_R f.$$

Remind that $\mathcal{B}_0 f = Q(\mu, f)$ and remark that x-derivatives commute with the operator $\mathcal{B}$, thus for any multi-index $\alpha,\beta\in\mathbb{N}^3$, we have

$$\partial^\alpha_v\partial^\beta_x(\mathcal{B} f) = \partial^\alpha_v(\mathcal{B}\partial^\beta_x f)$$

and

$$\partial^\alpha_v \mathcal{B}_0 f = \partial^\alpha_v Q(\mu,f) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} Q(\partial^{\alpha_1}_v\mu, \partial^{\alpha_2}_v f)$$

and, writing $v\cdot\nabla_x f = v_i\partial_{x_i} f$,

$$\partial^\alpha_v \mathcal{B} f = \mathcal{B}\partial^\alpha_v f + \sum_{\alpha_1+\alpha_2=\alpha, |\alpha_1|\ge 1} C_{\alpha_1,\alpha_2}\{Q(\partial^{\alpha_1}_v\mu, \partial^{\alpha_2}_v f) - (\partial^{\alpha_1}_v v_i)\partial_{x_i}(\partial^{\alpha_2}_v f) - M(\partial^{\alpha_1}_v\chi_R)(\partial^{\alpha_2}_v f)\}$$

finally

$$
\begin{aligned}
&\partial_v^\alpha \partial_x^\beta \mathcal{B} f = \mathcal{B}(\partial_v^\alpha \partial_x^\beta f) \\
&+ \sum_{\alpha_1+\alpha_2=\alpha, |\alpha_1|=1} C_{\alpha_1,\alpha_2} \{ Q(\partial_v^{\alpha_1}\mu, \partial_v^{\alpha_2}\partial_x^\beta f) - (\partial_v^{\alpha_1} v_i)\partial_{x_i}(\partial_v^{\alpha_2}\partial_x^\beta f) - M(\partial_v^{\alpha_1}\chi_R)(\partial_v^{\alpha_2}\partial_x^\beta f)\} \\
&+ \sum_{\alpha_1+\alpha_2=\alpha, |\alpha_1|\geq 2} C_{\alpha_1,\alpha_2} \{ Q(\partial_v^{\alpha_1}\mu, \partial_v^{\alpha_2}\partial_x^\beta f) - M(\partial_v^{\alpha_1}\chi_R)(\partial_v^{\alpha_2}\partial_x^\beta f)\}.
\end{aligned}
$$

We shall treat in full details the case $\ell = n = 1$, the others $\ell, n \geq 2$ being treated in the same way.

Case $\ell = n = 1$ **:** *Step 1. Derivatives in x.* First, using the computation (3.33) for $p = 1$, we have

$$
\frac{d}{dt}\|f\|_{L^1_{x,v}(m)} = \int \{\varphi_{m,1}(v) - M\chi_R(v)\}\, |f|\, m. \tag{3.35}
$$

As explained before, the x-derivatives commute with the operator $\mathcal{B}$, so for any multi-index $\beta \in \mathbb{N}^3$ we get from (3.33) that

$$
\frac{d}{dt}\|\partial_x^\beta f\|_{L^1_{x,v}(m)} = \int \{\varphi_{m,1}(v) - M\chi_R(v)\}|\partial_x^\beta f|\, m. \tag{3.36}
$$

Step 2. Derivatives in v. We now consider the derivatives in v. For any $\alpha \in \mathbb{N}^3$ with $|\alpha| = 1$, we compute the evolution of v-derivatives:

$$
\partial_t(\partial_v^\alpha f) = \mathcal{B}(\partial_v^\alpha f) + Q(\partial_v^\alpha \mu, f) - (\partial_v^\alpha v_i)\partial_{x_i} f - M(\partial_v^\alpha \chi_R) f.
$$

From the previous equation we deduce that

$$
\begin{aligned}
\frac{d}{dt}\|\partial_v^\alpha f\|_{L^1_{x,v}(m)} &= \int \{\mathcal{B}(\partial_v^\alpha f) + Q(\partial_v^\alpha \mu, f) - (\partial_v^\alpha v_i)\partial_{x_i} f - M(\partial_v^\alpha \chi_R) f\}\mathrm{sign}(\partial_v^\alpha f)\, m \\
&=: T_1 + T_2 + T_3 + T_4 + T_5,
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= \int \mathcal{B}(\partial_v^\alpha f)\, \mathrm{sign}(\partial_v^\alpha f)\, m \\
T_2 &= \int (\partial_v^\alpha \bar{a}_{ij})\, \partial_{ij} f\, \mathrm{sign}(\partial_v^\alpha f)\, m \\
T_3 &= -\int (\partial_v^\alpha \bar{c})\, f\, \mathrm{sign}(\partial_v^\alpha f)\, m \\
T_4 &= -\int (\partial_v^\alpha v_i)\partial_{x_i} f\, \mathrm{sign}(\partial_v^\alpha f)\, m = 0 \\
T_5 &= -\int M(\partial_v^\alpha \chi_R) f\, \mathrm{sign}(\partial_v^\alpha f)\, m.
\end{aligned}
$$

Again using the computation (3.33) of Lemma 3.2.50 for $p = 1$, we have

$$
T_1 = \int \{\varphi_{m,1}(v) - M\chi_R(v)\}|\partial_v^\alpha f|\, m.
$$

Concerning T_5, we use the following fact on the derivative of χ_R:

$$|\,\partial_v^\alpha \chi_R(v)| = \frac{1}{R}\left|\,\partial_v^\alpha \chi\left(\frac{v}{R}\right)\right| \le \frac{C}{R}\,\mathbf{1}_{R\le|v|\ \le 2R},$$

which implies that

$$T_5 \le M\frac{C}{R}\|\mathbf{1}_{R\le|v|\ \le 2R}\, f\|_{L^1_{x,v}(m)}.$$

Performing integration by parts, we get

$$T_2 + T_3 = -\int \partial_v^\alpha \bar a_{ij}\,\partial_i f\,\partial_j m\,\mathrm{sign}(\partial_v^\alpha f) + \int \partial_v^\alpha \bar b_j\,\partial_j m\, f\,\mathrm{sign}(\partial_v^\alpha f) =: A + B.$$

When m is a polynomial weight $m = \langle v\rangle^k$, we can easily estimate $T_2 + T_3$, thanks to another integration by parts, by

$$T_2 + T_3 = \int \{(\partial_v^\alpha \bar a_{ij})\,\partial_{ij} m + 2(\partial_v^\alpha \bar b_j)\,\partial_j m\}\, f\,\mathrm{sign}(\partial_v^\alpha f) \lesssim \|\langle v\rangle^{\gamma-1} f\|_{L^1_{x,v}(m)},$$

where we have used $|\partial_v^\alpha \bar a_{ij}| \le C\langle v\rangle^{\gamma+1}$, $|\partial_v^\alpha \bar b_j| \le \langle v\rangle^{\gamma}$, $|\partial_j m| \le C\langle v\rangle^{-1} m$ and $|\partial_{ij} m| \le C\langle v\rangle^{-2} m$.

We now investigate the case of (stretched) exponential weight $m = e^{r\langle v\rangle^s}$. First, we can easily estimate the term B, since $\partial_j m = C v_j \langle v\rangle^{\sigma-2} m$, as

$$B \lesssim \|\langle v\rangle^{\gamma+s-1} f\|_{L^1_{x,v}(m)}.$$

For the other term, integrating by parts again (first with respect to the ∂_v^α-derivative then to the ∂_i-derivative), gives us

$$A = -\int \left\{\bar a_{ij}\frac{\partial_{ij} m}{m} + \bar b_j \frac{\partial_j m}{m}\right\}\, |\partial_v^\alpha f|\, m + \int \bar a_{ij}\,\partial_i(\partial_v^\alpha m)\,\partial_j f\,\mathrm{sign}(\partial_v^\alpha f),$$

and we investigate the last term in the right-hand side. Recall that

$$\bar a_{ij}\xi_i\xi_j = \ell_1(v)|P_v\xi|^2 + \ell_2(v)|(I - P_v)\xi|^2,$$

we decompose $\partial_j f = P_v\partial_j f + (I - P_v)\partial_j f$ and similarly for $\partial_j(\partial_v^\alpha m)$, then a tedious but straightforward computation yields

$$\begin{aligned}\int \bar a_{ij}\,\partial_i(\partial_v^\alpha m)\,\partial_j f\,\mathrm{sign}(\partial_v^\alpha f) = \int &\{rs\ell_1(v)\langle v\rangle^{s-2} + rs(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} \\ &+ r^2 s^2 \ell_1(v)|v|^2\langle v\rangle^{2s-4}\}\, P_v\partial_v^\alpha f\,\mathrm{sign}(\partial_v^\alpha f)\, m \\ &+ \int rs\ell_2(v)\langle v\rangle^{s-2}\,(I - P_v)\partial_v^\alpha f\,\mathrm{sign}(\partial_v^\alpha f)\, m.\end{aligned}$$

Recall that $\varphi_{m,1}(v) = \bar a_{ij}\frac{\partial_{ij} m}{m} + 2\bar b_j \frac{\partial_j m}{m}$ (see eq. (3.22)), hence we obtain

$$T_1 + A \le \int \{\psi_{m,1}(v) - M\chi_R(v)\}\, |\partial_v^\alpha f|\, m$$

with

$$\psi_{m,1}(v) := \bar{b}_j \frac{\partial_j m}{m} + rs\ell_2(v)\langle v\rangle^{s-2} + rs\ell_1(v)\langle v\rangle^{s-2} + rs(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} + r^2s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4}.$$

Thanks to the asymptotic behaviour of $\ell_1(v)$ and $\ell_2(v)$ in Lemma 3.2.47 and arguing as in Lemma 3.2.49, we obtain first that

$$\begin{cases} \limsup_{|v|\to+\infty} \psi_{m,1}(v) \le -rs\langle v\rangle^{\gamma+s}, & \text{if } 0 < s < 2; \\ \limsup_{|v|\to+\infty} \psi_{m,1}(v) \le -2r(1-4r), & \text{if } s = 2; \end{cases} \tag{3.37}$$

and then for any positive $\lambda < \lambda_{m,1}$ and $\delta \in (0, \lambda_{m,1} - \lambda)$ we can choose M, R large enough such that $\psi_{m,1}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}$.

Putting together all the previous estimates of this step, and denoting $\varphi^\sigma(v) = \varphi_{m,1}(v)$ when $m = \langle v\rangle^k$ and $\varphi^\sigma(v) = \psi_{m,1}(v)$ when $m = e^{r\langle v\rangle^s}$, we obtain

$$\frac{d}{dt}\|\partial_v^\alpha f\|_{L^1_{x,v}(m)} \le \int \{\varphi^\sigma(v) - M\chi_R(v)\}\, |\partial_v^\alpha f|\, m + \int \left\{C\langle v\rangle^{\gamma+\sigma-1} + C\frac{M}{R}\mathbf{1}_{R\le|v|\le 2R}\right\} |f|\, m. \tag{3.38}$$

Step 3. Conclusion. Consider the standard norm on $W^{1,1}_{x,v}(m)$

$$\|f\|_{W^{1,1}_{x,v}(m)} = \|f\|_{L^1_{x,v}(m)} + \|\nabla_x f\|_{L^1_{x,v}(m)} + \|\nabla_v f\|_{L^1_{x,v}(m)}.$$

Gathering the previous estimates (3.35), (3.36) and (3.38), we finally obtain

$$\frac{d}{dt}\|f\|_{W^{1,1}_{x,v}(m)} \le \int \left\{\varphi_{m,1}(v) + C\langle v\rangle^{\gamma+\sigma-1} + M\,\frac{C}{R}\mathbf{1}_{R\le|v|\ \le 2R} - M\chi_R\right\} |f|\, m + \int \{\varphi_{m,1}(v) - M\chi_R\}|\nabla_x f|\, m + \int \{\varphi^\sigma(v) - M\chi_R\}|\nabla_v f|\, m.$$

Remark that, since $\sigma \in [0,2]$, the function $\phi^0_m(v) := \varphi_{m,1}(v) + C\langle v\rangle^{\gamma+\sigma-1}$ has the same asymptotic behaviour of $\varphi_{m,1}(v)$ (see eq. (3.25) and eq. (3.28)). Then, arguing as in Lemma 3.2.49 (and (3.37)), for any positive $\lambda < \lambda_{m,1}$ and $\delta \in (0, \lambda_{m,1} - \lambda)$, one may find $M > 0$ and $R > 0$ large enough such that

$$\varphi_{m,1}(v) + C\langle v\rangle^{\gamma+\sigma-1} + M\,\frac{C}{R}\mathbf{1}_{R\le|v|\ \le 2R} - M\chi_R \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma},$$
$$\varphi_{m,1}(v) - M\chi_R \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma},$$
$$\varphi^\sigma(v) - M\chi_R \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

This implies that

$$\frac{d}{dt}\|f\|_{W^{1,1}_{x,v}(m)} \le -\lambda\|f\|_{W^{1,1}_{x,v}(m)} - \delta\|f\|_{W^{1,1}_{x,v}(m\langle v\rangle^{\gamma+\sigma})},$$

which concludes the proof in the case $\ell = 1$.

Case $\ell \geq 2$ **:** The higher order derivatives are treated in the same way, so we omit the proof. $\square$

Lemma 3.2.52. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, $\ell \in \mathbb{N}$ *and* $n \in \mathbb{N}^*$. *Then, for any* $\lambda < \lambda_{m,2}$, *we can choose* $M > 0$ *and* $R > 0$ *large enough such that the operator* $\mathcal{B} + \lambda$ *is hypo-dissipative in* $H^n_x H^\ell_v(m)$, *in the sense that*

$$\forall\, t \geq 0, \quad \|S_{\mathcal{B}}(t)\|_{\mathscr{B}(H^n_x H^\ell_v(m))} \leq C e^{-\lambda t}.$$

Proof of Lemma 3.2.52. Let us consider the equation $\partial_t f = \mathcal{B} f = \mathcal{B}_0 f - M\chi_R f$. Again we treat the case $\ell = 1$ in full details, the others $\ell \geq 2$ being the same.

Case $\ell = n = 1$ **:** *Step 1.* L^2 *estimate.* The $L^2_{x,v}(m)$ estimate is a special case of Lemma 3.2.50, from which we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_{x,v}(m)} \leq -c_0 \int \{\langle v\rangle^\gamma |P_v \nabla_v f|^2 + \langle v\rangle^{\gamma+2} |(I - P_v)\nabla_v f|^2\}\, m^2 \\ + \int \{\varphi_{m,2}(v) - M\chi_R(v)\} f^2\, m^2. \end{aligned} \tag{3.39}$$

Step 2. x-derivatives. Recall that the x-derivatives commute with the equation, so for any $\beta \in \mathbb{N}^3$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial^\beta_x f\|^2_{L^2_{x,v}(m)} \leq -c_0 \int \{\langle v\rangle^\gamma |P_v \nabla_v (\partial^\beta_x f)|^2 + \langle v\rangle^{\gamma+2} |(I - P_v)\nabla_v (\partial^\beta_x f)|^2\}\, m^2 \\ + \int \{\varphi_{m,2}(v) - M\chi_R(v)\} |\partial^\beta_x f|^2\, m^2. \end{aligned} \tag{3.40}$$

Step 3. v-derivatives. Let $\alpha \in \mathbb{N}^3$ with $|\alpha| = 1$. We recall the equation satisfied by $\partial^\alpha_v f$

$$\partial_t \partial^\alpha_v f = \mathcal{B}(\partial^\alpha_v f) + Q(\partial^\alpha_v \mu, f) - (\partial^\alpha_v v_i)\, \partial_{x_i} f - M(\partial^\alpha_v \chi_R) f.$$

From last equation we deduce that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial^\alpha_v f\|^2_{L^2_{x,v}(m)} &= \int \{\mathcal{B}(\partial^\alpha_v f) + Q(\partial^\alpha_v \mu, f) - (\partial^\alpha_v v_i)\, \partial_{x_i} f - M(\partial^\alpha_v \chi_R) f\}\ \partial^\alpha_v f\, m^2 \\ &=: T_1 + T_2 + T_3 + T_4 + T_5, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \int \mathcal{B}(\partial^\alpha_v f)\, \partial^\alpha_v f\, m^2 \\ T_2 &= \int (\partial^\alpha_v \bar{a}_{ij})\, \partial_{ij} f\, \partial^\alpha_v f\, m^2 \\ T_3 &= -\int (\partial^\alpha_v \bar{c})\, f\, \partial^\alpha_v f\, m^2 \\ T_4 &= -\int (\partial^\alpha_v v_i)\, \partial_{x_i} f\, \partial^\alpha_v f\, m^2 \\ T_5 &= -\int M(\partial^\alpha_v \chi_R) f\, \partial^\alpha_v f\, m^2. \end{aligned}$$

We have from Lemma 3.2.50

$$\begin{aligned}T_1 \leq -c_0 \int \{\langle v\rangle^\gamma |P_v \nabla_v(\partial_v^\alpha f)|^2 + \langle v\rangle^{\gamma+2} |(I-P_v)\nabla_v(\partial_v^\alpha f)|^2\}\, m^2 \\ + \int \{\varphi_{m,2}(v) - M\chi_R(v)\} |\partial_v^\alpha f|^2\, m^2.\end{aligned} \tag{3.41}$$

The terms T_3, T_4 and T_5 are easy to estimate: for any $\varepsilon > 0$ we get

$$T_4 \leq \varepsilon \|\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\partial_x^\alpha f\|^2_{L^2_{x,v}(m)}, \tag{3.42}$$

$$T_5 \leq M\frac{C}{R}\|\mathbf{1}_{R\leq |v|\ \leq 2R}\, \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + M\frac{C}{R}\|\mathbf{1}_{R\leq |v|\ \leq 2R}\, f\|^2_{L^2_{x,v}(m)}, \tag{3.43}$$

and using Lemma 3.2.47-(b),

$$\begin{aligned}T_3 &\leq C\int \langle v\rangle^{\gamma-1}\, |f|\, |\partial_v^\alpha f|\, m^2 \\ &\leq C\|\langle v\rangle^{\frac{\gamma-1}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + C\|\langle v\rangle^{\frac{\gamma-1}{2}} f\|^2_{L^2_{x,v}(m)}.\end{aligned} \tag{3.44}$$

Let us now deal with the part T_2. Performing integrations by parts, we have:

$$\begin{aligned}T_2 &= \int (\partial_v^\alpha \bar a_{ij})\, \partial_{ij} f\, \partial_v^\alpha f\, m^2 \\ &= -\int (\partial_v^\alpha \bar b_j)\, \partial_j f\, \partial_v^\alpha f\, m^2 - \int (\partial_v^\alpha \bar a_{ij})\, \partial_j f\, \partial_i(\partial_v^\alpha f)\, m^2 - \int (\partial_v^\alpha \bar a_{ij})\, \partial_j f\, \partial_v^\alpha f\, \partial_i m^2 \\ &=: -\left(T_{21} + T_{22} + T_{23}\right).\end{aligned}$$

We first deal with T_{21}. Using Lemma 3.2.47, we have

$$\begin{aligned}T_{21} &\leq C\int \langle v\rangle^\gamma\, |\partial_j f|\, |\partial_v^\alpha f|\, m^2 \\ &\leq C\|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v f\|^2_{L^2_{x,v}(m)} = C\|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2_{x,v}(m)} + C\|\langle v\rangle^{\frac{\gamma}{2}} (I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)}.\end{aligned} \tag{3.45}$$

As far as T_{22} is concerned, the integration by parts gives,

$$\begin{aligned}T_{22} &= -\int \partial_v^\alpha\left[(1-\chi)m^2\right] \bar a_{ij}\, \partial_j f\, \partial_i(\partial_v^\alpha f) - \int (1-\chi)m^2\, \bar a_{ij}\, \partial_j(\partial_v^\alpha f)\, \partial_i(\partial_v^\alpha f) \\ &\quad - \int (1-\chi)m^2\, \bar a_{ij}\, \partial_j f\, \partial_i(\partial_v^\alpha \partial_v^\alpha f) - \int (\partial_v^\alpha \bar a_{ij})\, \partial_j f\, \partial_i(\partial_v^\alpha f)\, \chi m^2 \\ &=: -\left(\widetilde T_{221} + \widetilde T_{222} + \widetilde T_{223}\right) + T_{220}.\end{aligned}$$

Let us estimate $\widetilde{T}_{222}+\widetilde{T}_{223}$, using integration by parts,

$$\begin{aligned}
&\widetilde{T}_{222}+\widetilde{T}_{223} \\
&= \int (1-\chi) m^2 \Big[\ell_1(v)\, P_v \nabla_v(\partial_v^\alpha \partial_v^\alpha f)\cdot P_v\nabla_v f + \ell_2(v)\,(I-P_v)\nabla_v(\partial_v^\alpha\partial_v^\alpha f)\cdot (I-P_v)\nabla_v f\Big] \\
&\quad + \int (1-\chi) m^2 \Big[\ell_1(v)\, P_v\nabla_v(\partial_v^\alpha f)\cdot P_v\nabla_v(\partial_v^\alpha f) \\
&\quad + \ell_2(v)\,(I-P_v)\nabla_v(\partial_v^\alpha f)\cdot (I-P_v)\nabla_v(\partial_v^\alpha f)\Big] \\
&= -\widetilde{T}_{221} - \int (\partial_v^\alpha \ell_1(v))\, P_v\nabla_v(\partial_v^\alpha f)\cdot P_v\nabla_v f\,(1-\chi)m^2 \\
&\quad - \int (\partial_v^\alpha \ell_2(v))\,(I-P_v)\nabla_v(\partial_v^\alpha f)\cdot (I-P_v)\nabla_v f\,(1-\chi)m^2 \\
&\quad - \int \left[\ell_1(v)-\ell_2(v)\right](I-P_v)\partial_v^\alpha(\partial_v^\alpha f)\,\frac{v\cdot\nabla_v f}{|v|^2}\,(1-\chi)m^2 \\
&\quad - \int \left[\ell_1(v)-\ell_2(v)\right](I-P_v)\nabla_v\partial_v^\alpha f\,\frac{v\cdot\nabla_v g}{|v|^2}\,(1-\chi)m^2 \\
&=: -\widetilde{T}_{221} + T_{221} + \ldots + T_{224}\,.
\end{aligned}$$

This means $T_{22} = T_{220}+T_{221}+\ldots+T_{224}$. In order to estimate T_{22}, we need to estimate T_{22i} for $i=0,\ldots,4$ (lemma 3.2.47 plays an important role in those estimates). First of all, we obtain

$$\begin{aligned}
T_{220} &\le C\int_{|v|\le 2}\langle v\rangle^{\gamma+1}|\nabla_v f|\,|\nabla_v(\partial_v^\alpha f)|\,|\chi| m^2 \\
&\le \varepsilon\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v f\|^2_{L^2_{x,v}(m)}
\end{aligned}$$

For T_{221}, we have

$$\begin{aligned}
T_{221} &\le C\int_{|v|\ge 1}\langle v\rangle^{\gamma-1}|P_v\nabla_v f|\,|P_v\nabla_v(\partial_v^\alpha f)|\, m^2 \\
&\le \varepsilon\|\langle v\rangle^{\frac{\gamma-1}{2}} P_v\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma-1}{2}} P_v\nabla_v f\|^2_{L^2_{x,v}(m)}\,.
\end{aligned}$$

For T_{222}, we have

$$\begin{aligned}
T_{222} &\le C\int_{|v|\ge 1}\langle v\rangle^{\gamma+1}|(I-P_v)\nabla_v f|\,|(I-P_v)\nabla_v(\partial_v^\alpha f)|\, m^2 \\
&\le \varepsilon\|\langle v\rangle^{\frac{\gamma+1}{2}}(I-P_v)\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+1}{2}}(I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)}.
\end{aligned}$$

For T_{223}, we obtain

$$\begin{aligned}
T_{223} &\le C\int_{|v|\ge 1}(\langle v\rangle^{\gamma-1}+\langle v\rangle^{\gamma+1})|\nabla_v f|\,|(I-P_v)\nabla_v(\partial_v^\alpha f)|\, m^2 \\
&\le \varepsilon\|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v f\|^2_{L^2_{x,v}(m)}.
\end{aligned}$$

Finally, for T_{224},

$$\begin{aligned} T_{224} &\le C \int_{|v|\ge 1} (\langle v\rangle^{\gamma-1} + \langle v\rangle^{\gamma+1}) |\nabla_v(\partial_v^\alpha f)| \, |(I-P_v)\nabla_v f| \, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)} \end{aligned}$$

This completes the estimate of T_{22} that we write, gathering previous bounds, as

$$\begin{aligned} T_{22} \le \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v(\partial_v^\alpha f)\|_{L^2_{x,v}(m)} + \varepsilon \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\partial_v^\alpha f)\|_{L^2_{x,v}(m)} \\ C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|_{L^2_{x,v}(m)}. \end{aligned} \tag{3.46}$$

Concerning T_{23}, we apply the same process as T_{22}: we first write

$$\begin{aligned} T_{23} = &- \int (\partial_v^\alpha \bar a_{ij}) \, \partial_j f \, \partial_i m^2 \, \chi g \\ &- \int \partial_v^\alpha \ell_1(v) \, P_v \nabla_v m^2 \cdot P_v \nabla_v f \, (1-\chi) \, \partial_v^\alpha f \\ &- \int \partial_v^\alpha \ell_2(v) \, (I-P_v)\nabla_v m^2 \cdot (I-P_v)\nabla_v f \, (1-\chi) \, \partial_v^\alpha f \\ &- \int [\ell_1(v) - \ell_2(v)] \, (I-P_v)\partial_v^\alpha m^2 \, \frac{v\cdot \nabla_v f}{|v|^2} \, (1-\chi) \, \partial_v^\alpha f \\ &- \int [\ell_1(v) - \ell_2(v)] \, (I-P_v)\partial_v^\alpha f \, \frac{v\cdot \nabla_v m^2}{|v|^2} \, (1-\chi) \, \partial_v^\alpha f \\ =&: T_{230} + ... + T_{234}. \end{aligned}$$

Note that $(I-P_v)\nabla_v m^2 = 0$, one can easily get $T_{232} = T_{233} = 0$. Let us estimate the other terms, by Lemma 3.2.47, we have

$$\begin{aligned} T_{230} &\le C \int_{|v|\le 2} \langle v\rangle^{\gamma+\sigma} \, |\nabla_v f| \, |\partial_v^\alpha f| \, |\chi| \, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v f\|^2_{L^2_{x,v}(m)} \end{aligned}$$

also

$$\begin{aligned} T_{231} &\le C \int_{|v|>1} \langle v\rangle^{\gamma+\sigma-2} \, |P_v\nabla_v f| \, |\partial_v^\alpha f| \, m^2 \\ &\le C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|^2_{L^2_{x,v}(m)} + \varepsilon \|\langle v\rangle^{\frac{\gamma+2\sigma-4}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)}, \end{aligned}$$

and

$$\begin{aligned} T_{234} &\le C \int_{|v|>1} (\langle v\rangle^{\gamma+\sigma-2} + \langle v\rangle^{\gamma+\sigma}) \, |(I-P_v)\nabla_v f| \, |\partial_v^\alpha f| \, m^2 \\ &\le C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)} + \varepsilon \|\langle v\rangle^{\frac{\gamma+2\sigma-2}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)}. \end{aligned}$$

Gathering previous inequalities we complete the estimate of T_{23}

$$\begin{aligned} T_{23} \le \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + \varepsilon \|\langle v\rangle^{\frac{\gamma+2\sigma-2}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} \\ + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)}. \end{aligned} \tag{3.47}$$

Using the fact that $1+\langle v\rangle^\gamma+\langle v\rangle^{\gamma+2\sigma-2}\lesssim\langle v\rangle^{\gamma+\sigma}$ and putting together (3.41) to (3.47) we get,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} &\leq -(c_0-\varepsilon)\int\left\{\langle v\rangle^\gamma|P_v\nabla_v(\partial_v^\alpha f)|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(\partial_v^\alpha f)|^2\right\}m^2\\ &+\int\left\{\varphi_{m,2}(v)+\varepsilon\langle v\rangle^{\gamma+\sigma}+C\langle v\rangle^{\gamma-1}+M\frac{C}{R}\mathbf{1}_{R\leq|v|\leq 2R}-M\chi_R(v)\right\}|\partial_v^\alpha f|^2\,m^2\\ &+C(\varepsilon)\int\left\{\langle v\rangle^\gamma|P_v\nabla_v f|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v f|^2\right\}m^2\\ &+\int\left\{C\langle v\rangle^{\gamma-1}+M\frac{C}{R}\mathbf{1}_{R\leq|v|\leq 2R}\right\}|f|^2\,m^2+C(\varepsilon)\|\partial_x^\alpha f\|^2_{L^2_{x,v}(m)}.\end{aligned}\tag{3.48}$$

Step 4. Conclusion in the case $\ell=n=1$. We now introduce the following norm on $H^1_xH^1_v(m)$

$$\|f\|^2_{\widetilde{H}^1(m)}:=\|f\|^2_{L^2_{x,v}(m)}+\|\nabla_x f\|^2_{L^2_{x,v}(m)}+\eta\,\|\nabla_v f\|^2_{L^2_{x,v}(m)},$$

which is equivalent to the standard $H^1_{x,v}(m)$-norm for any $\eta>0$. Gathering estimates (3.39), (3.40) and (3.48) of previous steps, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_{\widetilde{H}^1(m)} &\leq (-c_0+\eta\,C(\varepsilon))\int\left\{\langle v\rangle^\gamma|P_v\nabla_v f|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v f|^2\right\}m^2\\ &+\int\left\{\psi^0_m(v)+\eta\,M\frac{C}{R}\,\mathbf{1}_{R\leq|v|\leq 2R}-M\chi_R(v)\right\}f^2\,m^2\\ &-c_0\sum_{|\beta|=1}\int\left\{\langle v\rangle^\gamma|P_v\nabla_v(\partial_x^\beta f)|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(\partial_x^\beta f)|^2\right\}m^2\\ &+\int\left\{\psi^1_m(v)-M\chi_R(v)\right\}|\nabla_x f|^2\,m^2\\ &+\eta(-c_0+\varepsilon)\sum_{|\alpha|=1}\int\left\{\langle v\rangle^\gamma|P_v\nabla_v(\partial_v^\alpha f)|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(\partial_v^\alpha f)|^2\right\}m^2\\ &+\eta\int\left\{\psi^2_m(v)+M\frac{C}{R}\,\mathbf{1}_{R\leq|v|\leq 2R}-M\chi_R(v)\right\}|\nabla_v f|^2\,m^2.\end{aligned}$$

where we have defined

$$\begin{aligned}\psi^0_m(v)&:=\varphi_{m,2}(v)+C\eta\langle v\rangle^{\gamma-1},\\ \psi^1_m(v)&:=\varphi_{m,2}(v)+\eta C(\varepsilon),\\ \psi^2_m(v)&:=\varphi_{m,2}(v)+\varepsilon\langle v\rangle^{\gamma+\sigma}+C\langle v\rangle^{\gamma-1}.\end{aligned}$$

Let us fix any $\lambda<\lambda_{m,2}$. We first choose $\varepsilon>0$ small enough so that $-c_0+\varepsilon<0$ and $-\lambda_{m,2}+\varepsilon<-\lambda$. Then we choose $\eta>0$ small enough such that $-c_0+\eta\,C(\varepsilon)\leq 0$ and $-\lambda_{m,2}+\eta C(\varepsilon)<-\lambda$. Hence the functions ψ^i_m have the same asymptotic behaviour than $\varphi_{m,2}$ (see (3.25), (3.28) and (3.30)). Then, using Lemma 3.2.49, for any $\lambda<\lambda_{m,2}$

and $\delta \in (0, \lambda_{m,2} - \lambda)$, one may find $M > 0$ and $R > 0$ large enough such that

$$\begin{aligned} \psi^0_m(v) + \eta\, M \frac{C}{R}\, \mathbf{1}_{R \le |v| \le 2R} - M\chi_R(v) &\le -\lambda - \delta \langle v \rangle^{\gamma+\sigma}, \\ \psi^1_m(v) - M\chi_R(v) &\le -\lambda - \delta \langle v \rangle^{\gamma+\sigma}, \\ \psi^2_m(v) + M \frac{C}{R}\, \mathbf{1}_{R \le |v|\ \le 2R} - M\chi_R(v) &\le -\lambda - \delta \langle v \rangle^{\gamma+\sigma}. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f\|^2_{\widetilde{H}^1(m)} \le -\lambda \|f\|^2_{\widetilde{H}^1(m)} &- \delta \|f\|^2_{\widetilde{H}^1(m\langle v \rangle^{(\gamma+\sigma)/2})} \\ &- K\Big\{ \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2(m)} + \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v) \nabla_v f\|^2_{L^2(m)} \Big\} \\ &- K\Big\{ \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v (\nabla_x f)\|^2_{L^2(m)} + \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v) \nabla_v (\nabla_x f)\|^2_{L^2(m)} \Big\} \\ &- K\Big\{ \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v (\nabla_v f)\|^2_{L^2(m)} + \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v) \nabla_v (\nabla_v f)\|^2_{L^2(m)} \Big\}, \end{aligned}$$

and then

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{H^1_{x,v}(m)} \le C e^{-\lambda t} \|f\|_{H^1_{x,v}(m)}.$$

This concludes the proof of the hypodissipativity of $\mathcal{B} + \lambda$ in $H^1_{x,v}(m)$.

Case $\ell \ge 2$: The higher order derivatives are treated in the same way, introducing the (equivalent) norm on $H^n_x H^\ell_v(m)$

$$\|f\|^2_{\widetilde{H^n_x H^\ell_v(m)}} = \|f\|^2_{L^2(m)} + \sum_{1 \le |\alpha| + |\beta| \le \max(\ell, n); |\alpha| \le \ell; |\beta| \le n} \eta^{|\alpha|}\, \|\partial^\alpha_v \partial^\beta_x f\|^2_{L^2(m)},$$

and choosing $\eta > 0$ small enough as in the case $\ell = 1$. □

Lemma 3.2.53. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, *$\ell \in \mathbb{N}$ and $n \in \mathbb{N}^*$, and $p \in [1, 2]$. Then, for any $\lambda < \lambda_{m,p}$, we can choose $M > 0$ and $R > 0$ large enough such that the operator $\mathcal{B} + \lambda$ is hypo-dissipative in $W^{n,p}_x W^{\ell,p}_v(m)$, in the sense that*

$$\forall\, t \ge 0, \quad \|S_{\mathcal{B}}(t)\|_{\mathscr{B}(W^{n,p}_x W^{\ell,p}_v(m))} \le C e^{-\lambda t}.$$

Proof. It is a consequence of Lemmas 3.2.51 and 3.2.52, together with the Riesz-Thorin interpolation theorem. □

Lemma 3.2.54. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**. *Then, for any $\lambda < \lambda_{m,2}$, we can choose M and R large enough such that the operator $\mathcal{B} + \lambda$ is hypo-dissipative in $H^n_x H^{-1}_v(m)$, for any $n \in \{-1\} \cup \mathbb{N}$, in the sense that*

$$\forall\, t \ge 0, \quad \|S_{\mathcal{B}}(t)\|_{\mathscr{B}(H^n_x H^{-1}_v(m))} \le C e^{-\lambda t}.$$

Proof. We consider the equation $\partial_t f = \mathcal{B} f$ and split the proof into five steps.

Step 1. We first make the change of unknown $h := fm$ and define the corresponding operator $\mathcal{B}_m h := m\,\mathcal{B}(m^{-1}h)$ which writes:

$$\begin{aligned}\mathcal{B}_m h &= m\,(a_{ij} * \mu)\partial_{ij}(m^{-1}h) - (c * \mu)h - v\cdot\nabla_x h - M\,\chi_R\,h \\ &= (m\,\partial_{ij}(m^{-1})(a_{ij}*\mu) - c*\mu - M\,\chi_R)h \\ &\quad + 2m\,\partial_j(m^{-1})(a_{ij}*\mu)\partial_i h - v\cdot\nabla_x h + (a_{ij}*\mu)\partial_{ij}h.\end{aligned}$$

We hence define $\mathcal{B}_m^*$, the (formal) adjoint operator of $\mathcal{B}_m$, by

$$\mathcal{B}_m^*\phi := \left(\frac{\partial_{ij}m}{m}\bar a_{ij} + 2\frac{\partial_j m}{m}\,\bar b_j - M\,\chi_R\right)\phi + 2\left(\bar b_i + \frac{\partial_j m}{m}\bar a_{ij}\right)\partial_i\phi + v\cdot\nabla_x\phi + \bar a_{ij}\partial_{ij}\phi.$$

Consequently, we have the estimate

$$\begin{aligned}\int (\mathcal{B}_m^*\phi)\,\phi &= \int\left(\bar a_{ij}\frac{\partial_{ij}m}{m} + 2\bar b_j\frac{\partial_j m}{m} - M\,\chi_R\right)\phi^2 \\ &\quad + \int\left(\bar a_{ij}\frac{\partial_j m}{m} + \bar b_i\right)\partial_i(\phi^2) + \int v\cdot\nabla_x\phi\,\phi + \int \bar a_{ij}\partial_{ij}\phi\,\phi \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Performing one integration by parts, we obtain

$$T_2 = \int\left(-\bar a_{ij}\frac{\partial_{ij}m}{m} + \bar a_{ij}\frac{\partial_i m}{m}\frac{\partial_j m}{m} - \bar b_j\frac{\partial_j m}{m} - \bar c\right)\phi^2.$$

The term T_3 gives no contribution thanks to its divergence structure in x. And we deal with T_4 using that $\partial_{ij}\phi\,\phi = \frac{1}{2}\partial_{ij}(\phi^2) - \partial_i\phi\partial_j\phi$, which implies

$$T_4 = -\int \bar a_{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}\int \bar c\,\phi^2.$$

Finally, we obtain that

$$\begin{aligned}\int \mathcal{B}_m^*\phi\,\phi &= -\int \bar a_{ij}\partial_i\phi\partial_j\phi + \int\{\tilde\varphi_{m,2} - M\chi_R\}\,\phi^2 \\ &\le -c_0\int\left\{\langle v\rangle^\gamma|P_v\nabla_v\phi|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v\phi|^2\right\} + \int\{\tilde\varphi_{m,2} - M\chi_R\}\,\phi^2.\end{aligned}$$

where we recall that $\tilde\varphi_{m,2}$ is defined in (3.23) and satisfies Lemma 3.2.49.

Thanks to Lemma 3.2.49, for any positive $\lambda < \lambda_{m,2}$ and $\delta \in (0, \lambda_{m,2} - \lambda)$, we can thus find M, R large enough such that $\tilde\varphi_{m,2}(v) - M\chi_R \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}$. We can conclude that

$$\begin{aligned}\int (\mathcal{B}_m^*\phi)\,\phi &\le -\lambda\|\phi\|_{L^2}^2 - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\phi\|_{L^2}^2 \\ &\quad - c_0\left\{\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v\phi\|_{L^2(m)}^2 + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v\phi\|_{L^2(m)}^2\right\}.\end{aligned}$$

Step 2. Since ∇_x commute with the operator $\mathcal{B}_m^*$, we can immediately obtain that if ϕ is solution of

$$\partial_t \phi = \mathcal{B}_m^* \phi, \tag{3.49}$$

we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla_x \phi\|^2_{L^2_{x,v}} \leq -c_0 \int \left(\langle v \rangle^\gamma |P_v \nabla_v (\nabla_x \phi)|^2 + \langle v \rangle^{\gamma+2} |(I - P_v)\nabla_v (\nabla_x \phi)|^2 \right) + \int \left(\tilde{\varphi}_{m,2}(v) - M\chi_R(v)\right) |\nabla_x \phi|^2.$$

Step 3. Now, we introduce the notation $\phi_\alpha := \partial_v^\alpha \phi$ where $\alpha \in \mathbb{N}^3$ and $|\alpha| = 1$. Let us write the equation satisfied by ϕ_α when ϕ is a solution of (3.49), we have

$$\partial_t \phi_\alpha = \mathcal{B}_m^* \phi_\alpha + \partial_v^\alpha \left\{ \bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_j \frac{\partial_j m}{m} \; - M\,\chi_R \right\} \phi + 2\partial_v^\alpha \left\{ \bar{a}_{ij} \frac{\partial_j m}{m} + \bar{b}_i \right\} \partial_i \phi + \partial_v^\alpha v \cdot \nabla_x \phi + \partial_v^\alpha \bar{a}_{ij} \partial_{ij} \phi,$$

which implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |\phi_\alpha|^2 &= \int (\mathcal{B}_m^* \phi_\alpha)\, \phi_\alpha + \int \partial_v^\alpha \left\{ \bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_j \frac{\partial_j m}{m} \; - M\,\chi_R \right\} \phi\, \phi_\alpha \\ &\quad + 2 \int \partial_v^\alpha \left\{ \bar{a}_{ij} \frac{\partial_j m}{m} + \bar{b}_i \right\} \partial_i \phi\, \phi_\alpha + \int (\partial_v^\alpha v_i)(\partial_{x_i} \phi)\, \phi_\alpha \\ &\quad + \int (\partial_v^\alpha \bar{a}_{ij})(\partial_{ij} \phi)\, \phi_\alpha \\ &=: T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

Using the step 1 of the proof, we have:

$$T_1 \leq -c_0 \int \left(\langle v \rangle^\gamma |P_v \nabla_v \phi_\alpha|^2 + \langle v \rangle^{\gamma+2} |(I - P_v)\nabla_v \phi_\alpha|^2 \right) + \int (\tilde{\varphi}_{m,2}(v) - M\chi_R)\, \phi_\alpha^2.$$

Concerning T_2, we have

$$\begin{aligned} T_2 &= \frac{1}{2} \int \partial_v^\alpha \left\{ \bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_j \frac{\partial_j m}{m} \; - M\,\chi_R \right\} \partial_v^\alpha (\phi^2) \\ &= -\frac{1}{2} \int \partial_v^\alpha \partial_v^\alpha \left\{ \bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_j \frac{\partial_j m}{m} \; - M\,\chi_R \right\} \phi^2 \\ &\leq \int \left(C\, \langle v \rangle^{\gamma+\sigma-2} + M \frac{C}{R^2} \mathbf{1}_{R \leq |v| \leq R^2} \right) \phi^2, \end{aligned}$$

where we have used the fact that $\bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_j \frac{\partial_j m}{m} \; \sim C \langle v \rangle^{\gamma+\sigma}$. Since $\partial_j m = C v_j \langle v \rangle^{\sigma-2} m$ we have

$$\partial_v^\alpha \left(\bar{a}_{ij} \frac{\partial_j m}{m} \right) = \partial_v^\alpha \left(\bar{a}_{ij}\, C v_j \langle v \rangle^{\sigma-2} \right) = C \partial_v^\alpha \left(v_i \ell_1(v) \langle v \rangle^{\sigma-2} \right)$$

which is of order $\langle v \rangle^{\gamma+\sigma-2}$. We hence deduce that also in this case, we have

$$T_3 \lesssim \int (\langle v \rangle^\gamma + \langle v \rangle^{\gamma+\sigma-2}) |\nabla_v \phi|^2.$$

Then, for any $\varepsilon > 0$, we have

$$T_4 \leq \varepsilon \int \phi_\alpha^2 + C(\varepsilon) \int |\partial_x^\alpha \phi|^2.$$

Finally, using the same method as in the proof of Lemma 3.2.52 (for the term T_2 in that lemma), we obtain

$$\begin{aligned} T_5 \leq \varepsilon \int & \left(\langle v \rangle^\gamma |P_v \nabla_v \phi_\alpha|^2 + \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v \phi_\alpha|^2 \right) \\ & + C(\varepsilon) \int \left(\langle v \rangle^\gamma |P_v \nabla_v \phi|^2 + \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v \phi|^2 \right). \end{aligned}$$

Step 4. We define the following norm on H^1

$$\|\phi\|^2_{\widetilde{H}^1_{x,v}} := \|\phi\|^2_{L^2_{x,v}} + \|\nabla_x \phi\|^2_{L^2_{x,v}} + \eta \, \|\nabla_v \phi\|^2_{L^2_{x,v}},$$

which is equivalent to the standard H^1-norm for any $\eta > 0$, and we compute its evolution when ϕ is a solution of (3.49). Gathering estimates of previous steps it follows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi\|^2_{\widetilde{H}^1_{x,v}} \leq & \, (-c_0 + \eta \, C(\varepsilon) + \eta C) \int \left(\langle v \rangle^\gamma |P_v \nabla_v \phi|^2 + \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v \phi|^2 \right) \\ & + \int \left(\tilde{\varphi}_{m,2}(v) + C \eta \langle v \rangle^{\gamma+\sigma-2} + \eta \, M \frac{C}{R} \, \mathbf{1}_{R \leq |v| \leq 2R} - M \chi_R(v) \right) \phi^2 \\ & - c_0 \int \left(\langle v \rangle^\gamma |P_v \nabla_v (\nabla_x \phi)|^2 + \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v (\nabla_x \phi)|^2 \right) \\ & + \int \left(\tilde{\varphi}_{m,2}(v) + \eta C(\varepsilon) - M \chi_R(v) \right) |\nabla_x \phi|^2 \\ & + \eta(-c_0 + \varepsilon C) \int \left(\langle v \rangle^\gamma |P_v \nabla_v (\nabla_v \phi)|^2 + \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v (\nabla_v \phi)|^2 \right) \\ & + \eta \int \left(\tilde{\varphi}_{m,2}(v) + C \varepsilon - M \chi_R(v) \right) |\nabla_v \phi|^2. \end{aligned}$$

We conclude as in Lemma 3.2.52: we first choose $\varepsilon > 0$ small enough and then $\eta > 0$ small enough, so that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi\|^2_{\widetilde{H}^1_{x,v}} \leq & \int \left(\tilde{\varphi}_{m,2}(v) + C \eta \langle v \rangle^{\gamma+\sigma-2} + \eta \, M \frac{C}{R} \, \mathbf{1}_{R \leq |v| \leq 2R} - M \chi_R(v) \right) \phi^2 \\ & + \int \left(\tilde{\varphi}_{m,2}(v) - M \chi_R(v) \right) |\nabla_x \phi|^2 \\ & + \eta \int \left(\tilde{\varphi}_{m,2}(v) + C \varepsilon - M \chi_R(v) \right) |\nabla_v \phi|^2. \end{aligned}$$

We deduce that for any positive $\lambda < \lambda_{m,2}$ and $\delta \in (0, \lambda_{m,2} - \lambda)$, one may find M and R such that

$$\frac{1}{2} \frac{d}{dt} \|\phi\|^2_{\widetilde{H}^1_{x,v}} \leq -\lambda \|\phi\|^2_{\widetilde{H}^1_{x,v}} - \delta \|\phi\|^2_{\widetilde{H}^1_{x,v}(\langle v \rangle^{(\gamma+\sigma)/2})}.$$

Step 5. We have proved that for any $\lambda < \lambda_{m,2}$,

$$\|S_{\mathcal{B}_m^*}(t)\phi\|^2_{H^1_{x,v}} \le C e^{-2\lambda t} \|\phi\|^2_{H^1_{x,v}} \quad \forall\, \phi \in H^1_{x,v}, \quad \forall\, t \ge 0.$$

The last inequality implies that for any $h \in H^{-1}_{x,v}$ and any $\phi \in H^1_{x,v}$,

$$\langle S_{\mathcal{B}_m}(t) h, \phi\rangle = \langle h, S_{\mathcal{B}_m^*}(t)\phi\rangle \le \|h\|_{H^{-1}_{x,v}} \|S_{\mathcal{B}_m^*}(t)\phi\|_{H^1_{x,v}} \le C e^{-\lambda t} \|h\|_{H^{-1}_{x,v}} \|\phi\|_{H^1_{x,v}}.$$

As a consequence, we obtain that

$$\|S_{\mathcal{B}_m}(t) h\|_{H^{-1}_{x,v}} \le C e^{-\lambda t} \|h\|_{H^{-1}_{x,v}}$$

and coming back to the operator $\mathcal{B}$,

$$\|S_{\mathcal{B}}(t) f\|_{H^{-1}_{x,v}(m)} \le C e^{-\lambda t} \|f\|_{H^{-1}_{x,v}(m)}.$$

Finally, using the following embeddings for any $n \in \mathbb{N}$,

$$H^n_x L^2_v(m) \subset H^n_x H^{-1}_v(m) \subset H^{-1}_x H^{-1}_v(m),$$

we deduce that the conclusion of Lemma 3.2.54 holds by interpolation (with the results from Lemma 3.2.52). □

3.2.5 Regularization

We now turn to the boundedness of $\mathcal{A}$ as well as regularization properties of $\mathcal{A}S_{\mathcal{B}}(t)$. We recall the operator $\mathcal{A}$ defined in (3.21)

$$\mathcal{A} f = \mathcal{A}_0 f + M\chi_R f = (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu + M\chi_R f,$$

for M and R large enough chosen before. Thanks to the smooth cut-off function χ_R, for any $q \in [1, +\infty]$, $p \ge q$ and any weight function m under the hypotheses **(H1)-(H2)-(H3)**, we easily obtain

$$\|M\chi_R f\|_{L^q_{x,v}(\mu^{-1/2})} \lesssim \|f\|_{L^q_x L^p_v(m)}.$$

Taking derivatives we get an analogous estimate, for any $n, \ell \in \mathbb{N}$,

$$\|M\chi_R f\|_{W^{n,q}_x W^{\ell,q}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,q}_x W^{\ell,p}_v(m)},$$

Arguing by duality we also have

$$\|M\chi_R f\|_{H^n_x H^{-1}_v(\mu^{-1/2})} \lesssim \|f\|_{H^n_x H^{-1}_v(m)}.$$

Finally we obtain

$$M\chi_R \in \begin{cases} \mathscr{B}\left(L^p_{x,v}(m), L^p_{x,v}(\mu^{-1/2})\right), & \forall\, p \in [1,\infty]; \\ \mathscr{B}\left(W^{n,p}_x W^{\ell,p}_v(m), W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})\right), & \forall\, p \in [1,2],\, n \in \mathbb{N}^*,\, \ell \in \mathbb{N}; \\ \mathscr{B}\left(H^n_x H^{-1}_v(m), H^n_x H^{-1}_v(\mu^{-1/2})\right), & \forall\, n \in \{-1\} \cup \mathbb{N}. \end{cases} \tag{3.50}$$

We know obtain the boundedness of $\mathcal{A}$.

Lemma 3.2.55. *Consider* **(H1)**, **(H2)** *or* **(H3)** *and a weight function m.*

(i) For any $p \in [1,\infty]$, there holds

$$\mathcal{A} \in \mathscr{B}\left(L^p_{x,v}(m), L^p_{x,v}(\mu^{-1/2})\right).$$

(ii) For any $p \in [1,2]$, $n \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, there holds

$$\mathcal{A} \in \mathscr{B}\left(W^{n,p}_x W^{\ell,p}_v(m), W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})\right).$$

(iii) For all $n \in \{-1\} \cup \mathbb{N}$, there holds

$$\mathcal{A} \in \mathscr{B}\left(H^n_x H^{-1}_v(m), H^n_x H^{-1}_v(\mu^{-1/2})\right).$$

In particular $\mathcal{A} \in \mathscr{B}(E) \cap \mathscr{B}(\mathcal{E})$ for any admissible space $\mathcal{E}$ in (3.19).

Proof. Thanks to (3.50) we just need to consider the operator $\mathcal{A}_0$. We write

$$\mathcal{A}_0 f = (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu$$

and split the proof into several steps.

Step 1. Since $\gamma \in [-2,1]$ we have $|a_{ij}(v-v_*)| \lesssim \langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+2}$, which implies $|(a_{ij} * f)(v)| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^1_v(\langle v\rangle^{\gamma+2})}$. Therefore, for any $p \in [1,\infty]$, we have

$$\|(a_{ij} * f)\partial_{ij}\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \|f\|_{L^1_v(\langle v\rangle^{\gamma+2})},$$

from which we can also easily deduce

$$\|\partial^\alpha_v \partial^\beta_x (a_{ij} * f)\partial_{ij}\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \sum_{\alpha_1 \le \alpha} \|\partial^{\alpha_1}_v \partial^\beta_x f\|_{L^1_v(\langle v\rangle^{\gamma+2})}.$$

Integrating in the x-variable, we finally get

$$\|(a_{ij} * f)\partial_{ij}\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,1}_v(\langle v\rangle^{\gamma+2})}.$$

Step 2. Assume $\gamma \in [0,1]$. In that case we have $|c(v-v_*)| \lesssim \langle v\rangle^{\gamma}\langle v_*\rangle^{\gamma}$ and the same argument as above gives

$$\|(c * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,1}_v(\langle v\rangle^{\gamma})}.$$

Step 3. Assume $\gamma \in [-2,0)$. We decompose $c = c_+ + c_-$ with $c_+ = c\mathbf{1}_{|\cdot|>1}$ and $c_- = c\mathbf{1}_{|\cdot|\le 1}$. For the non-singular term c_+ we easily get, for any $p \in [1,\infty]$,

$$\|(c_+ * f)\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \|f\|_{L^1_v}$$

whence

$$\|(c_+ * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,1}_v}.$$

We now investigate the singular term c_-. For any $p \in [1, 3/|\gamma|)$ we get

$$\begin{aligned}\|(c_- * f)\mu\|^p_{L^p_v(\mu^{-1/2})} = \|(c_- * f)\mu^{1/2}\|^p_{L^p_v} &\lesssim \int_v \left| \int_{v_*} |v - v_*|^\gamma \, \mathbf{1}_{|v-v_*|\le 1} \, |f(v_*)| \right|^p \mu^{1/2}(v) \\ &\lesssim \int_{v_*} |f(v_*)|^p \left\{ \int_v |v - v_*|^{\gamma p} \, \mathbf{1}_{|v-v_*|\le 1} \, \mu^{1/2}(v) \right\} \\ &\lesssim \|f\|^p_{L^p_v(\langle v \rangle^\gamma)},\end{aligned}$$

where we have used that $|\gamma| p < 3$ (so that the integral in v is bounded) and Lemma 3.3.59. Taking derivatives and integrating in x it follows

$$\|(c_- * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,p}_v(\langle v \rangle^\gamma)}, \quad \forall \, p \in [1, 3/|\gamma|).$$

Remark that by Hölder's inequality, for any $q \in (3/(3+\gamma), \infty]$ we have

$$\begin{aligned}|(c_- * f)(v)| &\lesssim \int_{v_*} |v - v_*|^\gamma \, \mathbf{1}_{|v-v_*|\le 1} \, |f(v_*)| \\ &\lesssim \left(\int_{v_*} |v - v_*|^{\gamma q'} \, \mathbf{1}_{|v-v_*|\le 1} \right)^{1/q'} \|f\|_{L^q_v} \lesssim \|f\|_{L^q_v},\end{aligned}$$

which implies

$$\|(c_- * f)\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \|f\|_{L^q_v}, \quad \forall \, p \in [1, \infty],$$

and similarly

$$\|(c_- * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,q}_v}, \quad \forall \, p \in [1, \infty].$$

Observe that in particular the operator $Tf = (c_- * f)\mu$ is a bounded operator from $W^{n,1}_x W^{\ell,1}_v(m) \to W^{n,1}_x W^{\ell,1}_v(\mu^{-1/2})$ and from $W^{n,\infty}_x W^{\ell,\infty}_v(m) \to W^{n,\infty}_x W^{\ell,\infty}_v(\mu^{-1/2})$, thus by interpolation also from $W^{n,p}_x W^{\ell,p}_v(m) \to W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})$ for any $p \in [1, \infty]$. This together with estimates of previous steps completes the proof of points (i) and (ii).

Step 4. We prove now (iii) by duality. We write the equality $\|(a_{ij} * f)\mu\|_{H^{-1}_{x,v}(\mu^{-1/2})} = \|(a_{ij} * f)\mu^{1/2}\|_{H^{-1}_{x,v}}$, hence we investigate $\sup_{\|\phi\|_{H^1_{x,v}} \le 1} |\langle (a_{ij} * f)\mu^{1/2}, \phi \rangle|$. We have for any $\theta > \gamma + 2 + 3/2$,

$$|\langle (a_{ij} * f)\mu^{1/2}, \phi \rangle| = |\langle \langle v \rangle^\theta f, \langle v \rangle^{-\theta} \{ a_{ij} * (\mu^{1/2}\phi) \} \rangle|$$

and

$$|(a_{ij} * \mu^{1/2}\phi)(v)| \lesssim \langle v \rangle^{\gamma+2} \, \|\mu^{1/2}\|_{L^2_v(\langle v \rangle^{\gamma+2})} \, \|\phi\|_{L^2_v}.$$

Therefore

$$|\langle \langle v \rangle^\theta f, \langle v \rangle^{-\theta} \{ a_{ij} * (\mu^{1/2}\phi) \} \rangle| \le \|\langle v \rangle^\theta f\|_{H^{-1}_{x,v}} \, \|\langle v \rangle^{-\theta} \{ a_{ij} * (\mu^{1/2}\phi) \}\|_{H^1_{x,v}}$$

with

$$\begin{aligned}\|\langle v \rangle^{-\theta} \{ a_{ij} * (\mu^{1/2}\phi) \}\|^2_{H^1_{x,v}} &\lesssim \|\langle v \rangle^{-\theta} \{ a_{ij} * (\mu^{1/2}\phi) \}\|^2_{L^2_{x,v}} + \|\nabla_v (\langle v \rangle^{-\theta}) \{ a_{ij} * (\mu^{1/2}\phi) \}\|^2_{L^2_{x,v}} \\ &\quad + \|\langle v \rangle^{-\theta} \{ a_{ij} * (\nabla_v \mu^{1/2} \phi + \mu^{1/2} \nabla_v \phi) \}\|^2_{L^2_{x,v}} + \|\langle v \rangle^{-\theta} \{ a_{ij} * (\mu^{1/2} \nabla_x \phi) \}\|^2_{L^2_{x,v}} \\ &\lesssim \|\langle v \rangle^{\gamma+2-\theta}\|^2_{L^2_v} \, \|\mu^{1/2}\|^2_{H^1_v(\langle v \rangle^{\gamma+2})} \, \|\phi\|^2_{H^1_{x,v}} \lesssim \|\phi\|^2_{H^1_{x,v}}\end{aligned}$$

and then

$$\|(a_{ij} * f)\mu\|_{H^{-1}_{x,v}(\mu^{-1/2})} \lesssim \|f\|_{H^{-1}_{x,v}(\langle v\rangle^\theta)}.$$

For the term $(c * f)\mu$ we argue in a similar way as in the previous step. □

We turn now to regularization properties of the semigroup $\mathcal{S}_\mathcal{B}$. We follow a technique introduced by Hérau [53] for Fokker-Plank equations (see also [95, Section A.21] and [64]).

Lemma 3.2.56. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *and let m_0 be some weight function with $\gamma + \sigma > 0$. Define*

$$m_1 := \begin{cases} m_0 & \text{if } \gamma \in [0,1]; \\ \langle v\rangle^{\frac{|\gamma|}{2}} m_0 & \text{if } \gamma \in [-2,0). \end{cases} \qquad m_2 := \begin{cases} m_0 & \text{if } \gamma \in [0,1]; \\ \langle v\rangle^{4|\gamma|} m_0 & \text{if } \gamma \in [-2,0). \end{cases}$$

Then there hold:

1. *from L^2 to H^ℓ for $\ell \geq 1$:*

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^2(m_1),H^\ell(m_0))} \leq C\, t^{-3\ell/2}$$

2. *from L^1 to L^2:*

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^1(m_2),L^2(m_1))} \leq C\, t^{-8}.$$

3. *from L^2 to L^∞:*

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^2(m_2),L^\infty(m_1))} \leq C\, t^{-8}.$$

4. *from H^{-1} to L^2:*

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(H^{-1}(m_1),L^2(m_0))} \leq C\, t^{-3/2}.$$

Proof of Lemma 3.2.56. We consider the equation $\partial_t f = \mathcal{B} f$ and split the proof into four steps.

Step 1: from L^2 to H^ℓ. We only prove the case $\ell = 1$, the other cases being treated in the same way. Let us define

$$\mathcal{F}(t,f) := \|f\|^2_{L^2(m_1)} + \alpha_1\, t\, \|\nabla_v f\|^2_{L^2(m_0)} + \alpha_2\, t^2\, \langle \nabla_x f, \nabla_v f\rangle_{L^2(m_0)} + \alpha_3\, t^3\, \|\nabla_x f\|^2_{L^2(m_0)}.$$

We now choose α_i, $i = 1,2,3$ such that $0 < \alpha_3 \leq \alpha_2 \leq \alpha_1 \leq 1$ and $\alpha_2^2 \leq 2\alpha_1\alpha_3$. Then, there holds

$$2\mathcal{F}(t,f) \geq \alpha_3\, t^3\, \|\nabla_{x,v} f\|^2_{L^2(m_0)}.$$

Moreover, denoting $f_t = \mathcal{S}_{\mathcal{B}}(t) f$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t,f_t) = \frac{d}{dt}\|f_t\|^2_{L^2(m_1)} + \alpha_1 \|\nabla_v f_t\|^2_{L^2(m_0)} + \alpha_1\, t\,\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(m_0)} \\ + 2\alpha_2\, t\,\langle \nabla_x f_t, \nabla_v f_t\rangle_{L^2(m_0)} + \alpha_2\, t^2\,\frac{d}{dt}\langle \nabla_x f_t, \nabla_v f_t\rangle_{L^2(m_0)} \\ + 3\alpha_3\, t^2\,\|\nabla_x f_t\|^2_{L^2(m_0)} + \alpha_3\, t^3\,\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(m_0)}.\end{aligned}$$

We need to compute

$$\frac{d}{dt}\langle \nabla_x f, \nabla_v f\rangle_{L^2(m_0)} = \sum_{|\alpha|=1}\int \{\partial_x^\alpha(\mathcal{B}f)\,(\partial_v^\alpha f) + (\partial_x^\alpha f)\,\partial_v^\alpha(\mathcal{B}f)\}\; m_0^2.$$

Let us denote $f_x := \partial_x^\alpha f$ and $f_v := \partial_v^\alpha f$ to simplify and recall that

$$\partial_x^\alpha(\mathcal{B}f) = \bar a_{ij}\partial_{ij} f_x - \bar c f_x - v\cdot\nabla_x f_x - M\chi_R f_x,$$

and

$$\begin{aligned}\partial_v^\alpha(\mathcal{B}f) = \bar a_{ij}\partial_{ij} f_v - \bar c f_v - v\cdot\nabla_x f_v - M\chi_R f_v \\ + (\partial_v^\alpha \bar a_{ij})\partial_{ij} f - (\partial_v^\alpha \bar c) f - f_x - M(\partial_v^\alpha \chi_R) f.\end{aligned}$$

Using the same computation as in Lemma 3.2.52, we obtain

$$\int \{\partial_x^\alpha(\mathcal{B}f)\,(\partial_v^\alpha f) + (\partial_x^\alpha f)\,\partial_v^\alpha(\mathcal{B}f)\}\; m_0^2 = T_0 + T_1 + T_2 + T_3,$$

where

$$T_0 := -2\int \bar a_{ij}\,\partial_i f_x\,\partial_j f_v\, m_0^2,$$

$$T_1 := \int \{\varphi_{m_0,2}(v) - 2M\chi_R(v)\}\, f_x\, f_v\, m_0^2,$$

$$\begin{aligned}T_2 := -\int \left\{(\partial_v^\alpha \bar a_{ij})\,\frac{\partial_i m_0^2}{m_0^2} + \partial_v^\alpha \bar b_j\right\}\partial_j f\, f_x\, m_0^2 - \int \{\partial_v^\alpha \bar c + M(\partial_v^\alpha\chi_R)\}\, f\, f_x\, m_0^2 \\ - \int |f_x|^2\, m_0^2\end{aligned}$$

and

$$T_3 := -\int (\partial_v^\alpha \bar a_{ij})\,\partial_i f\,\partial_j f_x\, m_0^2.$$

For the term T_1, from the proof of Lemma 3.2.49 we get

$$T_1 \lesssim \int \langle v\rangle^{\gamma+\sigma}|f_x|\,|f_v|\, m_0^2 \lesssim \varepsilon t\,\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\,\partial_x^\alpha f\|^2_{L^2(m_0)} + \varepsilon^{-1}t^{-1}\,\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\,\partial_v^\alpha f\|^2_{L^2(m_0)}.$$

In a similar way, using $|\partial_v^\alpha \bar a_{ij}| \le C\langle v\rangle^{\gamma+1}$, $|\partial_v^\alpha \bar b_j| \le C\langle v\rangle^{\gamma}$ and $|\partial_i m^2| \le C\langle v\rangle^{\sigma-1} m^2$, we obtain for the second term

$$\begin{aligned}
T_2 &\lesssim \int \langle v\rangle^{\gamma+\sigma} |\nabla_v f|\,|f_x|\, m_0^2 + \int \left\{ \langle v\rangle^{\gamma-1} + \frac{M}{R}\mathbf{1}_{R\le|v|\le 2R} \right\} |f|\,|f_x|\, m_0^2 - \|\partial_x^\alpha f\|^2_{L^2(m_0)} \\
&\lesssim \varepsilon t \int \left\{ \langle v\rangle^{\gamma+\sigma} + \langle v\rangle^{\gamma-1} + \frac{M}{R}\mathbf{1}_{R\le|v|\le 2R} \right\} |\partial_x^\alpha f|^2\, m_0^2 \\
&\quad + \varepsilon^{-1} t^{-1} \int \left\{ \langle v\rangle^{\gamma-1} + \frac{M}{R}\mathbf{1}_{R\le|v|\le 2R} \right\} |f|^2\, m_0^2 \\
&\quad + \varepsilon^{-1} t^{-1} \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} \nabla_v f\|^2_{L^2(m_0)} - \|\partial_x^\alpha f\|^2_{L^2(m_0)}.
\end{aligned}$$

We now investigate T_0 and, decomposing $\partial_i f_x = P_v \partial_i f_x + (I-P_v)\partial_i f_x$ and the same for $\partial_j f_v$, we easily get

$$\begin{aligned}
T_0 &\lesssim \varepsilon t \{ \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)} \} \\
&\quad + \varepsilon^{-1} t^{-1} \{ \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v(\partial_v^\alpha f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\partial_v^\alpha f)\|^2_{L^2(m_0)} \}.
\end{aligned}$$

For the remainder term T_3, arguing as in the proof of Lemma 3.2.52 (term T_{22} in that lemma, see (3.46)) gives us

$$\begin{aligned}
T_3 &\lesssim \varepsilon t \{ \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)} \} \\
&\quad + \varepsilon^{-1} t^{-1} \{ \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2(m_0)} \}.
\end{aligned}$$

Finally, putting together previous estimates we obtain

$$\begin{aligned}
&\int \{ \nabla_x(\mathcal{B}f)\nabla_v f + \nabla_x f \nabla_v(\mathcal{B}f) \}\, m_0^2 \\
&\lesssim \varepsilon t \Big\{ \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} \nabla_x f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v(\nabla_x f)\|^2_{L^2(m_0)} \\
&\quad + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\nabla_x f)\|^2_{L^2(m_0)} \Big\} + C\varepsilon^{-1} t^{-1} \Big\{ \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} \nabla_v f\|^2_{L^2(m_0)} \\
&\quad + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v(\nabla_v f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\nabla_v f)\|^2_{L^2(m_0)} \Big\} \\
&\quad + C\varepsilon^{-1} t^{-1} \Big\{ \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2(m_0)} \Big\} \\
&\quad + C\varepsilon^{-1} t^{-1} \|f\|^2_{L^2(m_0)} - \|\nabla_x f\|^2_{L^2(m_0)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we also write the following

$$2\alpha_2 t \langle \nabla_x f, \nabla_v f\rangle_{L^2(m_0)} \le \alpha_2 \left(\varepsilon t^2 \|\nabla_x f\|^2_{L^2(m_0)} + C\varepsilon^{-1} \|\nabla_v f\|^2_{L^2(m_0)} \right).$$

Moreover, picking up estimates of Lemma 3.2.52, it follows that: for any $0 < \lambda < \lambda_{m,2}$ and $0 < \delta < \lambda_{m,2} - \lambda$, there are $M, R > 0$ large enough such that,

$$\begin{aligned}
\int (\mathcal{B}f) f\, m_1^2 &\le -c_0 \Big\{ \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2(m_1)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2(m_1)} \Big\} \\
&\quad - \lambda \|f\|^2_{L^2(m_1)} - \delta \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f\|^2_{L^2(m_1)},
\end{aligned}$$

also, for some $\varepsilon_0 > 0$ to be chosen later,

$$\begin{aligned}\int \nabla_v(\mathcal{B}f)\nabla_v f\, m_0^2 \le &-c_0\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_v f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_v f)\|^2_{L^2(m_0)}\Big\}\\ &-\lambda\|\nabla_v f\|^2_{L^2(m_0)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_v f_t\|^2_{L^2(m_0)}\\ &+C\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v f\|^2_{L^2(m_0)}\Big\}\\ &+C\|f\|^2_{L^2(m_0)} + C\varepsilon_0^{-1}t^{-1}\|\nabla_v f\|^2_{L^2(m_0)} + C\varepsilon_0 t\|\nabla_x f\|^2_{L^2(m_0)},\end{aligned}$$

and finally

$$\begin{aligned}\int \nabla_x(\mathcal{B}f)\nabla_x f\, m_0^2 \le &-c_0\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_x f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_x f)\|^2_{L^2(m_0)}\Big\}\\ &-\lambda\|\nabla_x f\|^2_{L^2(m_0)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_x f\|^2_{L^2(m_0)}.\end{aligned}$$

We choose

$$\varepsilon_0 = \varepsilon^2, \quad \alpha_1 := \varepsilon^{5/2}, \quad \alpha_2 := \varepsilon^4, \quad \alpha_3 := \varepsilon^{9/2}.$$

Therefore, for any $t \in [0,1]$, we can gather previous estimates to obtain

$$\begin{aligned}&\frac{d}{dt}\,\mathcal{F}(t,f_t)\\ &\le \left(-c_0 + C\varepsilon^{1/2} + C\varepsilon^{5/2} + C\varepsilon^3\right)\left\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v f_t\|^2_{L^2(m_1)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v f_t\|^2_{L^2(m_1)}\right\}\\ &+t\varepsilon^{5/2}\left(-c_0 + C\varepsilon^{1/2}\right)\left\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_v f_t)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_v f_t)\|^2_{L^2(m_0)}\right\}\\ &+t^3\varepsilon^{9/2}\left(-c_0 + C\varepsilon^{1/2}\right)\left\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_x f_t)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_x f_t)\|^2_{L^2(m_0)}\right\}\\ &-\lambda\|f_t\|^2_{L^2(m_1)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)} + Ct(\varepsilon^{5/2}+\varepsilon^3)\|f_t\|^2_{L^2(m_0)}\\ &-\lambda\varepsilon^{5/2}t\|\nabla_v f_t\|^2_{L^2(m_0)} - t\varepsilon^{5/2}\left(\delta - C\varepsilon^{1/2}\right)\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_v f_t\|^2_{L^2(m_0)}\\ &-t^2\left(\lambda\varepsilon^{9/2}t - C\varepsilon^{9/2} - \varepsilon^5 - C\varepsilon^{9/2}t + \varepsilon^4\right)\|\nabla_x f_t\|^2_{L^2(m_0)}\\ &\quad -t^3\varepsilon^{9/2}\left(\delta - \varepsilon^{1/2}\right)\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_x f_t\|^2_{L^2(m_0)}.\end{aligned}$$

We then choose $\varepsilon > 0$ small enough such that the following conditions are fulfilled:

$$\begin{cases} -c_0 + C\varepsilon^{1/2} + C\varepsilon^{5/2} + C\varepsilon^3 < -K < 0,\\ -c_0 + C\varepsilon^{1/2} < -K < 0,\\ -\lambda + Ct(\varepsilon^{5/2}+\varepsilon^3) < -K < 0,\\ \delta - C\varepsilon^{1/2} < -K < 0,\\ C\varepsilon^{9/2} + \varepsilon^5 + C\varepsilon^{9/2} - \varepsilon^4 < -K < 0.\end{cases}$$

We have then proved that, for any $t \in [0,1]$,

$$\frac{d}{dt}\mathcal{F}(t,f_t) \le -K'\Big\{\|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^2\|\nabla_x f\|^2_{L^2(m_0)}\Big\} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)},$$

which implies

$$Ct^3 \| \nabla_{x,v} f_t\|^2_{L^2(m_0)} \le \mathcal{F}(t,f_t) \le \mathcal{F}(0,f_0) = \|f_0\|^2_{L^2(m_1)}.$$

We deduce

$$\forall\, t \in (0,1], \quad \| \nabla_{x,v} \mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(m_0)} \le C\, t^{-3/2}\, \|f_0\|_{L^2(m_1)},$$

and the proof of point (1) for $\ell = 1$ is complete.

Step 2: from L^1 to L^2. We define,

$$\begin{aligned}
\mathcal{G}(t,f_t) &:= \|f_t\|^2_{L^1(m_2)} + \alpha_0\, t^N \widetilde{\mathcal{F}}(t,f_t),\\
\widetilde{\mathcal{F}}(t,f_t) &:= \|f_t\|^2_{L^2(m_1)} + \alpha_1\, t^2 \|\nabla_v f_t\|^2_{L^2(m_0)}\\
&\qquad + \alpha_2\, t^4 \langle \nabla_x f_t, \nabla_v f_t\rangle_{L^2(m_0)} + \alpha_3\, t^6 \|\nabla_x f_t\|^2_{L^2(m_0)},
\end{aligned}$$

for some N to be chosen later. Thanks to Hölder and Sobolev inequalities (in $\mathbb{T}^3_x \times \mathbb{R}^3_v$), there holds

$$\|\langle v\rangle^q g\|^2_{L^2} \lesssim \|\nabla_{x,v} g\|^{3/2}_{L^2}\, \|\langle v\rangle^{4q} g\|^{1/2}_{L^1},$$

which implies that

$$\begin{aligned}
\|f\|^2_{L^2(m_1)} &\lesssim \|f\|^{1/2}_{L^1(m_2)}\, \|\nabla_{x,v}(m_0 f)\|^{3/2}_{L^2}\\
&\lesssim C_\varepsilon t^{-15} \|f\|^2_{L^1(m_2)} + \varepsilon t^5 \|\nabla_{x,v} f\|^2_{L^2(m_0)} + \varepsilon t^5 \|\langle v\rangle^{\sigma-1} f\|^2_{L^2(m_0)}\\
&\lesssim C_\varepsilon t^{-15} \|f\|^2_{L^1(m_2)} + \varepsilon t^5 \|\nabla_{x,v} f\|^2_{L^2(m_0)} + \varepsilon t^5 \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f\|^2_{L^2(m_1)},
\end{aligned} \tag{3.51}$$

where we have used in last line that $\langle v\rangle^{\sigma-1} m_0 \lesssim \langle v\rangle^{\frac{\gamma+\sigma}{2}} m_1$. Arguing as in step 1, we have

$$\frac{d}{dt}\widetilde{\mathcal{F}}(t,f_t) \le -K' \Big\{ \|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4 \|\nabla_x f\|^2_{L^2(m_0)} \Big\} - \delta \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)}.$$

Putting together previous estimates it follows

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}(t,f_t) &\le -K \|f_t\|^2_{L^1(m_2)} + \alpha_0 N t^{N-1} \widetilde{\mathcal{F}}(t,f)\\
&\quad - K' \alpha_0 t^N \Big\{ \|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4 \|\nabla_x f\|^2_{L^2(m_0)} \Big\}\\
&\quad - \delta \alpha_0 t^N \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)}\\
&\le -K \|f_t\|^2_{L^1(m_2)} + \alpha_0 N t^{N-1} \|f_t\|^2_{L^2(m_1)}\\
&\quad + C\alpha_0 N t^{N+1} \|\nabla_v f_t\|^2_{L^2(m_0)} + C\alpha_0 N t^{N+5} \|\nabla_x f_t\|^2_{L^2(m_0)}\\
&\quad - K' \alpha_0 t^N \Big\{ \|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4 \|\nabla_x f\|^2_{L^2(m_0)} \Big\}\\
&\quad - \delta \alpha_0 t^N \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)}.
\end{aligned}$$

Choose $t_* \in (0,1)$ so that $Nt^{N+1} \ll K' t^N$ then, for any $t \in [0,t_*]$,

$$\frac{d}{dt}\mathcal{G}(t,f_t) \le -K\|f_t\|^2_{L^1(m_2)} + C\alpha_0 t^{N-1}\|f_t\|^2_{L^2(m_1)} - \delta\alpha_0 t^N \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)} - K''\alpha_0 t^N\Big\{\|\nabla_v f_t\|^2_{L^2(m_0)} + t^4\|\nabla_x f\|^2_{L^2(m_0)}\Big\}.$$

Thanks to (3.51), for any $t \in [0,t_*]$, we get

$$\frac{d}{dt}\mathcal{G}(t,f_t) \le -(K - C_\varepsilon \alpha_0 t^{N-16})\|f_t\|^2_{L^1(m_2)} - \alpha_0 t^N(\delta - C\varepsilon)\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)} - \alpha_0 t^{N+4}(K'' - C\varepsilon)\|\nabla_{x,v} f\|^2_{L^2(m_0)}$$

Taking $N = 16$ and choosing $\varepsilon > 0$ small enough then $\alpha_0 > 0$ small enough, we get $\frac{d}{dt}\mathcal{G}(t,f_t) \le 0$ then

$$\forall\, t \in [0,t_*], \quad Ct^{16}\|f_t\|^2_{L^2(m_1)} \le \mathcal{G}(t,f_t) \le \mathcal{G}(0,f_0) = \|f_0\|^2_{L^1(m_2)}.$$

This ends the proof of point (2), using the fact that the norm is propagated for $t > t_*$.

Step 4: From L^2 to L^∞. Arguing by duality as in Lemma 3.2.54, the proof follows as in step 2.

Step 5: From H^{-1} to L^2. Using the duality approach as in Lemma 3.2.54, the proof follows arguing as in step 1. □

Corollary 3.2.57. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, *and spaces $\mathcal{E}_0, \mathcal{E}_1$ of the type E or $\mathcal{E}$ defined in* (3.18) *and* (3.19)*. Then for any $\lambda' < \lambda < \lambda_{m,p}$, there exists $N \in \mathbb{N}$ such that*

$$\|(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)}(t)\|_{\mathscr{B}(\mathcal{E}_1,\mathcal{E}_0)} \le C\, e^{-\lambda' t}, \qquad \forall\, t \ge 0.$$

Proof. It is a consequence of the hypodissipativity properties of $\mathcal{B}$ (Lemmas 3.2.50, 3.2.51, 3.2.52, 3.2.53 and 3.2.54), the boundedness of the operator $\mathcal{A}$ (Lemma 3.2.55), and the regularization properties in Lemma 3.2.56, together with [64, Lemma 2.4] and [51, Lemma 2.17]. □

3.2.6 Proof of Theorem 3.2.45

Thanks to the estimates proven in previous section, we can now turn to the proof of Theorem 3.2.45.

Proof of Theorem 3.2.45. Let $\mathcal{E}$ be an admissible space defined in (3.19) and consider $\ell_0 \ge 1$ large enough such that $E := H^{\ell_0}_{x,v}$ defined in (3.18) satisfies $E \subset \mathcal{E}$. Recall that in the small/reference space E we already have a spectral gap in Theorem 3.1.44.

Then the proof of Theorem 3.2.45 is a consequence of the hypo-dissipative properties of $\mathcal{B}$ in Lemmas 3.2.50, 3.2.51, 3.2.52, 3.2.53, 3.2.54, the boundedness of $\mathcal{A}$ in Lemma 3.2.55 and the regularizing properties of $(\mathcal{A} * \mathcal{S}_\mathcal{B})^{(*N)}(t)$ in Corollary 3.2.57, with which we are able to apply the "extension theorem" from [51, Theorem 2.13] and [64, Theorem 1.1]. □

3.3 The nonlinear equation

This section is devoted to the proof of Theorem 3.1.42. We develop a perturbative Cauchy theory for the (nonlinear) Landau equation using the estimates on the linearized operator obtained in the previous section.

Hereafter we consider hypothesis **(H0)** and some weight function m.

3.3.1 Functional spaces

We recall the following definitions

$$\|f\|^2_{H^1_{v,*}(m)} = \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2_v(m)},$$

and we also define the (stronger) norm

$$\|f\|^2_{H^1_{v,**}(m)} = \|\langle v\rangle^{\frac{\gamma+2}{2}} f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2_v(m)}.$$

We define the space $H^{-1}_{v,*}(m)$ by duality

$$\|f\|_{H^{-1}_{v,*}(m)} = \sup_{\|h\|_{H^1_{v,*}(m)}\le 1} \langle f, h\rangle_{L^2_v(m)}.$$

Hence, we can define the space $\mathcal{H}^3_x H^{-1}_{v,*}(m)$ associated to the norm

$$\|f\|^2_{\mathcal{H}^3_x H^{-1}_{v,*}(m)} = \|f\|^2_{L^2_x H^{-1}_{v,*}(m)} + \|\nabla_x f\|^2_{L^2_x H^{-1}_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} + \|\nabla^2_x f\|^2_{L^2_x H^{-1}_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} + \|\nabla^3_x f\|^2_{L^2_x H^{-1}_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.$$

Observe that $\mathcal{H}^3_x L^2_v(m)$ and $\mathcal{H}^3_x H^{-1}_{v,*}(m)$ can be seen as interpolation spaces of some admissible spaces $\mathcal{E}$ in (3.19). Therefore the exponential decay for the semigroup $\mathcal{S}_\Lambda(t)$ of the linearized Landau equation in Theorem 3.2.45 also holds in $\mathcal{H}^3_x L^2_v(m)$ and $\mathcal{H}^3_x H^{-1}_{v,*}(m)$.

3.3.2 Dissipative norm for the linearized equation

We constuct now a norm for which the linearized semigroup $\mathcal{S}_\Lambda(t)$ is dissipative, with a rate as close as we want to the optimal rate decay from Theorem 3.2.45, and also has a stronger dissipativity property.

Proposition 3.3.58. *Let $X := \mathcal{H}^3_x L^2_v(m)$ and $Y := \mathcal{H}^3_x H^1_{v,*}(m)$, and consider some weight function m' satisfying* **(H1)-(H2)-(H3)** *with $m' \lesssim m\langle v\rangle^{-(1-\sigma/2)}$. Define for any $\eta > 0$ and any $\lambda_2 < \lambda_1$ (where $\lambda_1 > 0$ is the optimal rate in Theorem 3.2.45) the equivalent norm*

$$|||f|||^2_{\mathcal{H}^3_x L^2_v(m)} := \eta \|f\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau. \tag{3.52}$$

Then there is $\eta > 0$ small enough such that the solution $\mathcal{S}_\Lambda(t) f$ to the linearized equation satisfies, for any $t \ge 0$ and some constant $K > 0$, $\forall\, f \in X$, $\Pi_0 f = 0$,

$$\frac{1}{2}\frac{d}{dt} |||\mathcal{S}_\Lambda(t) f|||^2_{\mathcal{H}^3_x L^2_v(m)} \le -\lambda_2 |||\mathcal{S}_\Lambda(t) f|||^2_{\mathcal{H}^3_x L^2_v(m)} - K \|\mathcal{S}_\Lambda(t) f\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)}.$$

Proof. First we remark that the norm $|||\cdot|||_{\mathcal{H}^3_x L^2_v(m)}$ is equivalent to the norm $\|\cdot\|_{\mathcal{H}^3_x L^2_v(m)}$ defined in (3.11) for any $\eta > 0$ and any $\lambda_2 < \lambda_1$. Indeed, using Theorem 3.2.45 (that also holds in $\mathcal{H}^3_x L^2_v(m)$), we have

$$\begin{aligned}\eta\|f\|^2_{\mathcal{H}^3_x L^2_v(m)} \le |||f|||^2_{\mathcal{H}^3_x L^2_v(m)} &= \eta\|f\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\ &\le \eta\|f\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty C^2 e^{-2(\lambda_1-\lambda_2)\tau} \|f\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \le (\eta + C)\|f\|^2_{\mathcal{H}^3_x L^2_v(m)}.\end{aligned}$$

We now compute, denoting $f_t = \mathcal{S}_\Lambda(t) f$,

$$\frac{1}{2}\frac{d}{dt}|||f_t|||^2_{\mathcal{H}^3_x L^2_v(m)} = \eta\langle \Lambda f_t\, f_t\rangle_{\mathcal{H}^3_x L^2_v(m)} + \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 t} f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau =: I_1 + I_2.$$

For I_1 we write $\Lambda = \mathcal{A}+\mathcal{B}$. Arguing exactly as in Section 3.2, more precisely Lemma 3.2.55, we first obtain that $\mathcal{A} \in \mathscr{B}(\mathcal{H}^3_x L^2_v(m), \mathcal{H}^3_x L^2_v(\mu^{-1/2}))$, whence

$$\langle \mathcal{A} f_t, f_t\rangle_{\mathcal{H}^3_x L^2_v(m)} \le C\|f_t\|_{\mathcal{H}^3_x L^2_v(m')}.$$

Moreover, repeating the estimates for the hypodissipativity of $\mathcal{B}$ in Lemmas 3.2.50 and 3.2.52 we easily get, for any $\lambda_2 \le \lambda < \lambda_{m,2}$ and some $K > 0$,

$$\langle \mathcal{B} f, f\rangle_{\mathcal{H}^3_x L^2_v(m)} \le -\lambda\|f\|^2_{\mathcal{H}^3_x L^2_v(m)} - K\|f\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)},$$

therefore it follows

$$I_1 \le -\lambda\eta\|f_t\|^2_{\mathcal{H}^3_x L^2_v(m)} - \eta K\|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} + \eta C\|f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}.$$

The second term is computed exactly

$$\begin{aligned} I_2 &= \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\|\mathcal{S}_\Lambda(\tau+t) f\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\ &= \frac{1}{2}\int_0^\infty \frac{\partial}{\partial \tau}\|\mathcal{S}_\Lambda(\tau+t) f\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau - \lambda_2\int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\ &= \frac{1}{2}\left[\|\mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}\right]_{\tau=0}^{\tau=+\infty} - \lambda_2\int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\ &= -\frac{1}{2}\|f_t\|^2_{\mathcal{H}^3_x L^2_v(m')} - \lambda_2\int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \end{aligned}$$

where we have used the semigroup decay.

Gathering previous estimates and using that $\lambda \ge \lambda_2$ we obtain

$$\begin{aligned} I_1 + I_2 \le -\lambda_2\left\{\eta\|f_t\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}\, d\tau\right\} \\ - \eta K\|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} + \eta C\|f_t\|^2_{\mathcal{H}^3_x L^2_v(m')} - \frac{1}{2}\|f_t\|^2_{\mathcal{H}^3_x L^2_v(m')}. \end{aligned}$$

We complete the proof choosing $\eta > 0$ small enough. □

3.3.3 Nonlinear estimates

We prove in this section some estimates for the nonlinear operator Q. We will use the following auxiliary results.

Lemma 3.3.59. *Let $-3 < \alpha < 0$ and $\theta > 3$. Then*

$$A_\alpha(v) := \int_{\mathbb{R}^3} |v - v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha.$$

Proof. Let $|v| \le 1/2$, thus $|v_*| + 1/2 \le 1 + |v - v_*|$ and we get

$$A_\alpha(v) = \int_{\mathbb{R}^3} |v_*|^\alpha \langle v - v_* \rangle^{-\theta} \, dv_* \lesssim \int_{\mathbb{R}^3} |v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha.$$

Consider now $|v| > 1/2$ and split the integral into two regions: $|v - v_*| > \langle v \rangle/4$ and $|v - v_*| \le \langle v \rangle/4$. For the first region we obtain

$$\int_{\mathbb{R}^3} \mathbf{1}_{|v-v_*| > \frac{\langle v \rangle}{4}} |v - v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha \int_{\mathbb{R}^3} \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha.$$

For the second region, $|v| > 1/2$ and $|v - v_*| \le \langle v \rangle/4$ imply $|v_*| \ge |v|/4$, hence

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathbf{1}_{|v-v_*| \le \frac{\langle v \rangle}{4}} |v - v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* &\lesssim \langle v \rangle^{-\theta} \int_{\mathbb{R}^3} \mathbf{1}_{|v-v_*| \le \frac{\langle v \rangle}{4}} |v - v_*|^\alpha \, dv_* \\
&\lesssim \langle v \rangle^{-\theta+\alpha+3} \lesssim \langle v \rangle^\alpha.
\end{aligned}$$

□

Lemma 3.3.60. *There holds:*

(i) For any $\theta > \gamma + 4 + 3/2$

$$|(a_{ij} * f)(v) \, v_i v_j| + |(a_{ij} * f)(v) \, v_i| + |(a_{ij} * f)(v)| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L^2_v(\langle v \rangle^\theta)}.$$

(ii) For any $\theta' > (\gamma+1)_+ + 3/2$ (where $x_+ := \max\{x, 0\}$)

$$|(b_j * f)(v)| \lesssim \langle v \rangle^{\gamma+1} \|f\|_{L^2_v(\langle v \rangle^{\theta'})}.$$

(iii) If $\gamma \in [0,1]$, for any $\theta'' > \gamma + 3/2$

$$|(c * f)(v)| \lesssim \langle v \rangle^{\gamma} \|f\|_{L^2_v(\langle v \rangle^{\theta''})}.$$

(iv) If $\gamma \in [-2, 0)$, for any $p > \frac{3}{3+\gamma}$ and $\theta'' > 3(1 - 1/p)$

$$|(c * f)(v)| \lesssim \langle v \rangle^{\gamma} \|f\|_{L^p_v(\langle v \rangle^{\theta''})}.$$

In particular, when $\gamma \in (-3/2, 0)$ we can choose $p = 2$ and $\theta'' > 3/2$; and when $\gamma \in [-2, -3/2]$ we can choose $p = 4$ and $\theta'' > 9/4$.

Proof. Recall that 0 is an eigenvalue of the matrix a_{ij} so that $a_{ij}(v-v_*)v_i = a_{ij}(v-v_*)v_{*i}$ and $a_{ij}(v-v_*)v_iv_j = a_{ij}(v-v_*)v_{*i}v_{*j}$. Using this we can easily obtain, for any $\theta > \gamma + 4 + 3/2$,

$$\begin{aligned} |(a_{ij} * f)(v)\, v_iv_j| &= |\int_{v_*} a_{ij}(v-v_*)v_iv_jf_*| = \left|\int_{v_*} a_{ij}(v-v_*)v_{*i}v_{*j}f_*\right| \\ &\lesssim \int_{v_*} \langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+4}|f_*| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^1_v(\langle v\rangle^{\gamma+4})} \\ &\lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v(\langle v\rangle^{\theta})}. \end{aligned}$$

In a similar way we get

$$|(a_{ij} * f)(v)\, v_i| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v(\langle v\rangle^{\theta-1})},$$

and we easily have, since $\gamma \in [-2, 1]$,

$$|(a_{ij} * f)(v)| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^2_v(\langle v\rangle^{\theta-2})}.$$

For the term $(b * f)$, we recall that $b_i(z) = -2|z|^{\gamma}z_i$ and we separate into two cases. When $\gamma \in [-1, 1]$ we have, for any $\theta' > \gamma + 1 + 3/2$,

$$\begin{aligned} |(b_i * f)(v)| &\lesssim \int_{v_*} |v-v_*|^{\gamma+1}\,|f_*| \lesssim \int_{v_*} \langle v\rangle^{\gamma+1}\langle v_*\rangle^{\gamma+1}|f_*| \\ &\lesssim \langle v\rangle^{\gamma+1}\|f\|_{L^1_v(\langle v\rangle^{\gamma+1})} \lesssim \langle v\rangle^{\gamma+1}\|f\|_{L^2_v(\langle v\rangle^{\theta'})}. \end{aligned}$$

When $\gamma \in [-2, -1)$ we use Lemma 3.3.59 to obtain, for any $\theta' > 3/2$,

$$\begin{aligned} |(b_i * f)(v)| &\lesssim \int_{v_*} |v-v_*|^{\gamma+1}\,\langle v_*\rangle^{-\theta'}\langle v_*\rangle^{\theta'}|f_*| \\ &\lesssim \left(\int_{v_*} |v-v_*|^{2(\gamma+1)}\,\langle v_*\rangle^{-2\theta'}\right)^{1/2}\|f\|_{L^2_v(\langle v\rangle^{\theta'})} \\ &\lesssim \langle v\rangle^{\gamma+1}\,\|f\|_{L^2_v(\langle v\rangle^{\theta'})}. \end{aligned}$$

Finally for the last term $(c * f)$, recall that $c(z) = -2(\gamma+3)|z|^{\gamma}$ and separate into two cases. When $\gamma \in [0, 1]$ then, for any $\theta'' > \gamma + 3/2$,

$$\begin{aligned} |(c * f)(v)| &\lesssim \int_{v_*} |v-v_*|^{\gamma}\,|f_*| \lesssim \int_{v_*} \langle v\rangle^{\gamma}\langle v_*\rangle^{\gamma}|f_*| \\ &\lesssim \langle v\rangle^{\gamma}\|f\|_{L^1_v(\langle v\rangle^{\gamma})} \lesssim \langle v\rangle^{\gamma}\|f\|_{L^2_v(\langle v\rangle^{\theta''})}. \end{aligned}$$

When $\gamma \in [-2, 0)$ we use Lemma 3.3.59 to obtain, for any $p > \frac{3}{3+\gamma}$ and for any $\theta'' > 3(1-1/p)$,

$$\begin{aligned} |(c * f)(v)| &\lesssim \int_{v_*} |v-v_*|^{\gamma}\,\langle v_*\rangle^{-\theta''}\langle v_*\rangle^{\theta''}|f_*| \\ &\lesssim \left(\int_{v_*} |v-v_*|^{\gamma\frac{p}{p-1}}\,\langle v_*\rangle^{-\theta''\frac{p}{p-1}}\right)^{(p-1)/p}\|f\|_{L^p_v(\langle v\rangle^{\theta''})} \\ &\lesssim \langle v\rangle^{\gamma}\,\|f\|_{L^2_v(\langle v\rangle^{\theta''})}, \end{aligned}$$

thanks to $|\gamma|p/(p-1) < 3$. ∎

We now prove nonlinear estimates for the Landau operator Q.

Lemma 3.3.61. *Consider hypothesis* **(H1)**, **(H2)** *or* (**H3**).

(i) For any $\theta > \gamma + 4 + 3/2$, *there holds*

$$\langle Q(f,g),h\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|_{H^1_{v,**}(m)} \|h\|_{H^1_{v,*}(m)}.$$

(ii) For any $\theta > \gamma + 4 + 3/2$ *and* $\theta' > 9/4$, *there holds*

$$\langle Q(f,g),g\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|^2_{H^1_{v,*}(m)}, \quad \text{if } \gamma \in (-3/2,1];$$

and if $\gamma \in [-2,-3/2]$,

$$\langle Q(f,g),g\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|^2_{H^1_{v,*}(m)} + \|f\|_{H^1_v(\langle v\rangle^{\theta'})} \|g\|^2_{L^2_v(m)}.$$

Proof. We write

$$\begin{aligned}\langle Q(f,g),h\rangle_{L^2_v(m)} &= \int \partial_j\{(a_{ij}*f)\partial_i g - (b_j*f)g\}\, h\, m^2 \\ &= -\int (a_{ij}*f)\partial_i g\, \partial_j h\, m^2 - \int (a_{ij}*f)\partial_i g\, \partial_j m^2\, h \\ &\quad + \int (b_j*f) g\, \partial_j h\, m^2 + \int (b_j*f) g\, h\, \partial_j m^2 \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Step 1. Point (i). We estimate each term separately.

Step 1.1. For the first term, since the estimate for $|v| \le 1$ is evident, we only consider the case $|v| > 1$. We decompose $\partial_i g = P_v \partial_i g + (I - P_v)\partial_i g$ and similarly for $\partial_j h$, where we recall that $P_v \partial_i g = v_i |v|^{-2}(v\cdot\nabla_v g)$. We hence write

$$\begin{aligned}T_1 &= \int (a_{ij}*f)\,\{P_v\partial_i g\, P_v\partial_j h + P_v\partial_i g\,(I-P_v)\partial_j h + (I-P_v)\partial_i g\, P_v\partial_j h \\ &\quad + (I-P_v)\partial_i g\,(I-P_v)\partial_j h\}\, m^2 \\ &=: T_{11} + T_{12} + T_{13} + T_{14}.\end{aligned}$$

Therefore we have, using Lemma 3.3.60,

$$\begin{aligned}T_{11} &= \int (a_{ij}*f) v_i v_j \,\frac{(v\cdot\nabla_v g)}{|v|^2}\,\frac{(v\cdot\nabla_v h)}{|v|^2}\, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+2} |v|^{-2}\, |\nabla_v g|\, |\nabla_v h|\, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v(m)} \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v(m)}.\end{aligned}$$

Moreover

$$\begin{aligned}T_{12} &= \int (a_{ij}*f) v_i \,\frac{(v\cdot\nabla_v g)}{|v|^2}\,\{(I-P_v)\partial_j h\}\, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+2} |v|^{-1}\, |\nabla_v g|\, |(I-P_v)\nabla_v h|\, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v(m)} \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v h\|_{L^2_v(m)},\end{aligned}$$

and similarly

$$T_{13} \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v g\|_{L^2_v(m)} \, \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v(m)}.$$

For the term T_{14} we obtain

$$\begin{aligned} T_{14} &= \int (a_{ij} * f) \, \{(I-P_v)\partial_i g\} \, \{(I-P_v)\partial_j h\} \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+2} |(I-P_v)\nabla_v g| \, |(I-P_v)\nabla_v h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v g\|_{L^2_v(m)} \, \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v h\|_{L^2_v(m)}. \end{aligned}$$

Step 1.2. Let us investigate the second term T_2, and again we only consider $|v| > 1$. Since $\partial_j m^2 = C v_j \langle v\rangle^{\sigma-2} m^2$, where we recall that $\sigma = 0$ when $m = \langle v\rangle^k$ and $\sigma = s$ when $m = e^{r\langle v\rangle^s}$, the same argument as for T_1 gives us

$$\begin{aligned} T_2 &= \int (a_{ij} * f) \, \{P_v \partial_i g \, \partial_j m^2 + (I-P_v)\partial_i g \, \partial_j m^2\} \, h \\ &=: T_{21} + T_{22}. \end{aligned}$$

Then we have

$$\begin{aligned} T_{21} &= C \int (a_{ij} * f) v_i v_j \langle v\rangle^{\sigma-2} \, \frac{(v \cdot \nabla_v g)}{|v|^2} \, h \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+2} \langle v\rangle^{\sigma-2} \, |v|^{-1} \, |\nabla_v g| \, |h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma+\sigma-2}{2}} \nabla_v g\|_{L^2_v(m)} \, \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} h\|_{L^2_v(m)}, \end{aligned}$$

and we recall that $\gamma + \sigma - 2 \le \gamma$. For the other term we get

$$\begin{aligned} T_{21} &= C \int (a_{ij} * f) v_j \langle v\rangle^{\sigma-2} \, \{(I-P_v)\partial_i g\} \, h \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+2} \langle v\rangle^{\sigma-2} \, |(I-P_v)\nabla_v g| \, |h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} (I-P_v)\nabla_v g\|_{L^2_v(m)} \, \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} h\|_{L^2_v(m)}, \end{aligned}$$

and recall that $\gamma + \sigma \le \gamma + 2$.

Step 1.3. For the term T_4,

$$\begin{aligned} T_4 &= C \int (b_j * f) \, v_j \langle v\rangle^{\sigma-2} \, g \, h \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+1} \langle v\rangle^{\sigma-1} \, |g| \, |h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} g\|_{L^2_v(m)} \, \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} h\|_{L^2_v(m)}. \end{aligned}$$

Remark that up to now we have obtained

$$T_1 + T_2 + T_4 \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|g\|_{H^1_{v,*}(m)} \, \|h\|_{H^1_{v,*}(m)},$$

however in the estimate of the term T_3 (see below) we will get a worst estimate (with the norm $\|g\|_{H^1_{v,**}(m)}$ instead of $\|g\|_{H^1_{v,*}(m)}$).

Step 1.4. We finally investigate the term T_3 and we get

$$\begin{aligned} T_3 &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \int \langle v\rangle^{\gamma+1} |g|\, |\nabla_v h|\, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma+2}{2}} g\|_{L^2_v(m)} \, \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v(m)} \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|g\|_{H^1_{v,**}(m)} \, \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v(m)}. \end{aligned}$$

We complete the proof of point (i) gathering previous estimates.

Step 2. Point (ii). Arguing as in Step 1, with h replaced by g, we already have

$$T_1 + T_2 + T_4 \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|g\|^2_{H^1_{v,*}(m)},$$

and we only estimate the term T_3. Integrating by parts we get

$$T_3 = \int (b_j * f)\, g\, \partial_j g\, m^2 \quad = -\frac{1}{2} \int (c * f)\, g^2\, m^2 - \frac{1}{2} \int (b_j * f)\, \partial_j m^2\, g^2 =: I + II.$$

The term II can be estimated exactly as T_4. For I, thanks to Lemma 3.3.60, we obtain

$$I \lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma}{2}} g\|^2_{L^2_v(m)}, \quad \text{if } \gamma \in (-3/2, 1];$$

and

$$\begin{aligned} I &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma}{2}} g\|^2_{L^2_v(m)} + \|f\|_{L^4_v(\langle v\rangle^{\theta'})} \, \|\langle v\rangle^{\frac{\gamma}{2}} g\|^2_{L^2_v(m)}, \quad \text{if } \gamma \in [-2, -3/2]; \\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^\theta)} \, \|\langle v\rangle^{\frac{\gamma}{2}} g\|^2_{L^2_v(m)} + \|f\|_{H^1_v(\langle v\rangle^{\theta'})} \, \|\langle v\rangle^{\frac{\gamma}{2}} g\|^2_{L^2_v(m)} \end{aligned}$$

and that concludes the proof. ☐

Lemma 3.3.62. *Let assumption* **(H0)** *be in force.*

(i) There holds

$$\langle Q(f,g), h\rangle_{\mathcal{H}^3_x L^2_v(m)} \lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)} \, \|g\|_{\mathcal{H}^3_x H^1_{v,**}(m)} \, \|h\|_{\mathcal{H}^3_x H^1_{v,*}(m)},$$

therefore

$$\|Q(f,g)\|_{\mathcal{H}^3_x H^{-1}_{v,*}(m)} \lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)} \, \|g\|_{\mathcal{H}^3_x H^1_{v,**}(m)}.$$

(ii) There holds

$$\langle Q(f,g), g\rangle_{\mathcal{H}^3_x L^2_v(m)} \lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)} \, \|g\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} \quad \textit{if } \gamma \in (-3/2, 1],$$

and

$$\begin{aligned} \langle Q(f,g), g\rangle_{\mathcal{H}^3_x L^2_v(m)} &\lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)} \, \|g\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} \\ &\quad + \|f\|_{\mathcal{H}^3_x H^1_{v,*}(m)} \, \|g\|^2_{\mathcal{H}^3_x L^2_v(m)} \quad \textit{if } \gamma \in [-2, -3/2]. \end{aligned}$$

Proof. We only prove point (ii). Point (i) can be proven in the same manner, using the estimate of Lemma 3.3.61-(i) instead of Lemma 3.3.61-(ii) as we shall do next.

We write

$$\langle Q(f,g),g\rangle_{\mathcal{H}^3_x L^2_v(m)} = \langle Q(f,g),g\rangle_{L^2_x L^2_v(m)} + \sum_{1\le|\beta|\le 3} \langle \partial^\beta_x Q(f,g), \partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-|\beta|(1-\sigma/2)})},$$

and

$$\partial^\beta_x Q(f,g) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2} Q(\partial^{\beta_1}_x f, \partial^{\beta_2}_x g).$$

Recall some inequalities that we shall use in the sequel:

$$\|u\|_{L^\infty(\mathbb{T}^3_x)} \lesssim \|u\|_{H^2(\mathbb{T}^3_x)}, \quad \|u\|_{L^6(\mathbb{T}^3_x)} \lesssim \|\nabla u\|_{L^2(\mathbb{T}^3_x)}, \quad \|u\|_{L^3(\mathbb{T}^3_x)} \lesssim \|\nabla u\|^{1/2}_{L^2(\mathbb{T}^3_x)} \|u\|^{1/2}_{L^2(\mathbb{T}^3_x)}.$$

Step 1. Using Lemma 3.3.61-(ii) we get

$$\begin{aligned}\langle Q(f,g),g\rangle_{L^2_x L^2_v(m)} &\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|^2_{H^1_{v,*}(m)} \quad \text{if } \gamma\in(-3/2,1], \\ &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|g\|^2_{L^2_x H^1_{v,*}(m)},\end{aligned}$$

and, similarly,

$$\begin{aligned}&\langle Q(f,g),g\rangle_{L^2_x L^2_v(m)} \\ &\quad\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|g\|^2_{L^2_x H^1_{v,*}(m)} + \|f\|_{H^2_x H^1_v(\langle v\rangle^{\theta'})} \|g\|^2_{L^2_x L^2_v(m)} \quad \text{if } \gamma\in[-2,-3/2].\end{aligned}$$

Step 2. Case $|\beta|=1$. From Lemma 3.3.61-(ii) it follows if $\gamma\in(-3/2,1]$,

$$\begin{aligned}&\langle Q(f,\partial^\beta_x g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|^2_{H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})}.\end{aligned}$$

and, similarly, if $\gamma\in[-2,-3/2]$,

$$\begin{aligned}\langle Q(f,\partial^\beta_x g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad + \|f\|_{H^2_x H^1_v(\langle v\rangle^{\theta'})} \|\nabla_x g\|^2_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})}.\end{aligned}$$

Moreover, thanks to Lemma 3.3.61-(i), we get

$$\begin{aligned}&\langle Q(\partial^\beta_x f, g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|_{H^1_{v,**}(m\langle v\rangle^{-(1-\sigma/2)})} \|\nabla_x g\|_{H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|g\|_{L^2_x H^1_{v,**}(m\langle v\rangle^{-(1-\sigma/2)})} \|\nabla_x g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})}.\end{aligned}$$

Step 3. Case $|\beta| = 2$. When $\beta_2 = \beta$, Lemma 3.3.61-(ii) yields if $\gamma \in (-3/2, 1]$,

$$\begin{aligned}\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} &\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x^2 g\|^2_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})}.\end{aligned}$$

and, similarly, if $\gamma \in [-2, -3/2]$,

$$\begin{aligned}\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad + \|f\|_{H^2_x H^1_v(\langle v\rangle^{\theta'})} \|\nabla^2_x g\|^2_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})}.\end{aligned}$$

If $|\beta_1| = |\beta_2| = 1$ then, thanks to Lemma 3.3.61-(i), we obtain

$$\begin{aligned}&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|_{H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla^2_x g\|_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|_{L^2_x H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla^2_x g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})}.\end{aligned}$$

Finally when $\beta_1 = \beta$, Lemma 3.3.61-(i) gives us

$$\begin{aligned}&\langle Q(\partial_x^\beta f, g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla^2_x f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|_{H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla^2_x g\|_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla^2_x f\|_{L^6_x L^2_v(\langle v\rangle^\theta)} \|g\|_{L^3_x H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla^2_x g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla^3_x f\|_{L^2_x L^2_v(\langle v\rangle^\theta)} \|g\|^{1/2}_{L^2_x H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla_x g\|^{1/2}_{L^2_x H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\qquad\quad \|\nabla^2_x g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})}.\end{aligned}$$

Step 4. Case $|\beta| = 3$. When $\beta_2 = \beta$, Lemma 3.3.61-(ii) implies if $\gamma \in (-3/2, 1]$,

$$\begin{aligned}\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})} &\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x^3 g\|^2_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})} \\ &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned}$$

and, similarly, if $\gamma \in [-2, -3/2]$,

$$\begin{aligned}\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})} &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla^3_x g\|^2_{L^2_x H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})} \\ &\quad + \|f\|_{H^2_x H^1_v(\langle v\rangle^{\theta'})} \|\nabla^3_x g\|^2_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned}$$

If $|\beta_1| = 1$ and $|\beta_2| = 2$ then, thanks to Lemma 3.3.61-(i), we obtain

$$\begin{aligned}&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|_{H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \|\nabla^3_x g\|_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|_{L^2_x H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \|\nabla^3_x g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned}$$

If $|\beta_1| = 2$ and $|\beta_2| = 1$ then, thanks to Lemma 3.3.61-(i), we obtain

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^{\beta} g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})} \\
&\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\theta})} \, \|\nabla_x g\|_{H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \, \|\nabla_x^3 g\|_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})} \\
&\quad\lesssim \|\nabla_x^3 f\|_{L^2_x L^2_v(\langle v\rangle^{\theta})} \, \|\nabla_x g\|^{1/2}_{L^2_x H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \\
&\qquad\quad \|\nabla_x^2 g\|^{1/2}_{L^2_x H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \, \|\nabla_x^3 g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.
\end{aligned}$$

Finally when $\beta_1 = \beta$, Lemma 3.3.61-(i) gives us

$$\begin{aligned}
&\langle Q(\partial_x^{\beta} f, g), \partial_x^{\beta} g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})} \\
&\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x^3 f\|_{L^2_v(\langle v\rangle^{\theta})} \, \|g\|_{H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \, \|\nabla_x^3 g\|_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})} \\
&\quad\lesssim \|\nabla_x^3 f\|_{L^2_x L^2_v(\langle v\rangle^{\theta})} \, \|g\|_{H^2_x H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \, \|\nabla_x^3 g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.
\end{aligned}$$

Step 5. Conclusion. We can conclude the proof gathering previous estimates and remarking that, for any $n = 0, 1, 2$, there holds

$$\|\langle v\rangle^{\frac{\gamma+2}{2}} \nabla_x^n g\|_{L^2_x L^2_v(m\langle v\rangle^{-(n+1)(1-\sigma/2)})} = \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} \nabla_x^n g\|_{L^2_x L^2_v(m\langle v\rangle^{-n(1-\sigma/2)})}$$

which implies

$$\|\nabla_x^n g\|_{L^2_x H^1_{v,**}(m\langle v\rangle^{-(n+1)(1-\sigma/2)})} \lesssim \|\nabla_x^n g\|_{L^2_x H^1_{v,*}(m\langle v\rangle^{-n(1-\sigma/2)})},$$

and observing also that

$$\|f\|_{H^3_x L^2_v(\langle v\rangle^{\theta})} \lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)}$$

and

$$\|f\|_{H^2_x H^1_v(\langle v\rangle^{\theta'})} \lesssim \|f\|_{\mathcal{H}^3_x H^1_{v,*}(m)} \quad \text{if } \gamma \in [-2, -3/2].$$

□

3.3.4 Proof of Theorem 3.1.42

We consider the Cauchy problem for the perturbation $f = F - \mu$. The equation satisfied by $f = f(t, x, v)$ is

$$\begin{cases} \partial_t f = \Lambda f + Q(f, f) \\ f_{|t=0} = f_0 = F_0 - \mu. \end{cases} \tag{3.53}$$

From the conservation laws (see (3.6) and (3.10)), for all $t > 0$, $\Pi_0 f_t = 0$ since $\Pi_0 f_0 = 0$, more precisely $\int f_t(v)\,dx\,dv = \int v_j f_t(v)\,dx\,dv = \int |v|^2 f_t(v)\,dx\,dv = 0$, and also $\Pi_0 Q(f_t, f_t) = 0$ because $\Pi_0 Q(f_0, f_0) = 0$.

Consider some weight function m and assumption **(H0)**. We split the proof of Theorem 3.1.42 into three parts: Theorem 3.3.64, Theorem 3.3.65 and Theorem 3.3.66 below.

Stability estimate

We start proving a stability estimate.

Proposition 3.3.63. *A solution $f = f_t$ to* (3.53) *satisfies, at least formally, the following differential inequality: for any $\lambda_2 < \lambda_1$ there holds*

$$\frac{1}{2}\frac{d}{dt}|||f|||^2_{\mathcal{H}^3_x L^2_v(m)} \le -\lambda_2 |||f|||^2_{\mathcal{H}^3_x L^2_v(m)} - \Big(K - C|||f|||_{\mathcal{H}^3_x L^2_v(m)}\Big) \|f\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)},$$

for some constants $K, C > 0$.

Proof of Proposition 3.3.63. Recall the norm $|||\cdot|||_{\mathcal{H}^3_x L^2_v(m)}$ is defined in Proposition 3.3.58. Thanks to (3.53) we write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|||f_t|||^2_{\mathcal{H}^3_x L^2_v(m)} &= \eta\langle f_t, \Lambda f_t\rangle_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \langle S_\Lambda(\tau)e^{\lambda_2\tau} f_t, S_\Lambda(\tau)e^{\lambda_2\tau}\Lambda f_t\rangle_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\
&\quad + \eta\langle f_t, Q(f_t,f_t)\rangle_{\mathcal{H}^3_x L^2_v(m)} \\
&\quad + \int_0^\infty \langle S_\Lambda(\tau)e^{\lambda_2\tau} f_t, S_\Lambda(\tau)e^{\lambda_2\tau} Q(f_t,f_t)\rangle_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the linear part $I_1 + I_2$, we already have from Propposition 3.3.58 that, for any $\lambda_2 < \lambda_1$,

$$I_1 + I_2 \le -\lambda_2 |||f_t|||^2_{\mathcal{H}^3_x L^2_v(m)} - K\|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)}.$$

Let us investigate the nonlinear part. For the term I_4, we use the fact that $\Pi_0 f_t = 0$ and $\Pi_0 Q(f_t,f_t) = 0$ for all $t \ge 0$, together with Theorem 3.2.45 to get

$$\begin{aligned}
&\int_0^\infty \langle S_\Lambda(\tau)e^{\lambda_2\tau} f_t, S_\Lambda(\tau)e^{\lambda_2\tau} Q(f_t,f_t)\rangle_{\mathcal{H}^3_x L^2_v(m')}\, d\tau \\
&\quad \le \int_0^\infty \|S_\Lambda(\tau)e^{\lambda_2\tau} f_t\|_{\mathcal{H}^3_x H^1_{v,*}(m')}\, \|S_\Lambda(\tau)e^{\lambda_2\tau} Q(f_t,f_t)\|_{\mathcal{H}^3_x H^{-1}_{v,*}(m')}\, d\tau \\
&\quad \le \|f_t\|_{\mathcal{H}^3_x H^1_{v,*}(m')}\, \|Q(f_t,f_t)\|_{\mathcal{H}^3_x H^{-1}_{v,*}(m')} \int_0^\infty C^2 e^{-2(\lambda_1-\lambda_2)\tau}\, d\tau \\
&\quad \lesssim \|f_t\|_{\mathcal{H}^3_x H^1_{v,*}(m')}\, \|Q(f_t,f_t)\|_{\mathcal{H}^3_x H^{-1}_{v,*}(m')}.
\end{aligned}$$

From Lemma 3.3.62-(i) we have

$$\|Q(f_t,f_t)\|_{\mathcal{H}^3_x H^{-1}_{v,*}(m')} \lesssim \|f_t\|_{\mathcal{H}^3_x L^2_v(m')}\, \|f_t\|_{\mathcal{H}^3_x H^1_{v,**}(m')}.$$

Therefore, using that $m' \lesssim m\langle v\rangle^{-(1-\sigma/2)}$ so that $\|f\|_{H^1_{v,**}(m')} \lesssim \|f\|_{H^1_{v,*}(m)}$, we obtain

$$I_4 \lesssim \|f_t\|_{\mathcal{H}^3_x L^2_v(m)}\, \|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} \lesssim |||f_t|||_{\mathcal{H}^3_x L^2_v(m)}\, \|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)}.$$

For the term I_3, Lemma 3.3.62-(ii) gives us directly

$$\begin{aligned}
I_3 &\lesssim \|f_t\|_{\mathcal{H}^3_x L^2_v(m)}\, \|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)} + \|f_t\|^2_{\mathcal{H}^3_x L^2_v(m)}\, \|f_t\|_{\mathcal{H}^3_x H^1_{v,*}(m)} \\
&\lesssim |||f_t|||_{\mathcal{H}^3_x L^2_v(m)}\, \|f_t\|^2_{\mathcal{H}^3_x H^1_{v,*}(m)}.
\end{aligned}$$

We complete the proof gathering previous bounds. □

Cauchy problem in the close-to-equilibrium setting

Consider **(H0)** and some weight m. We fix some weight function m_0 satisfying **(H1)-(H2)-(H3)** such that $m_0 \lesssim m\langle v\rangle^{-(1-\sigma/2)}$, which is always possible. We will construct solutions on $L^\infty_t(\mathcal{H}^3_x L^2_v(m))$ under a smallness assumption on the initial data $\|f_0\|_{\mathcal{H}^3_x L^2_v(m)} \le \epsilon_0$. Introduce the notation to simplify

$$\begin{cases} X := \mathcal{H}^3_x L^2_v(m), \quad Y := \mathcal{H}^3_x H^1_{v,*}(m), \\ X_0 := \mathcal{H}^3_x L^2_v(m_0), \quad Y_0 := \mathcal{H}^3_x H^1_{v,*}(m_0), \quad Z_0 := \mathcal{H}^3_x H^1_{v,**}(m_0) \end{cases}$$

and remark that $\|f\|_{Z_0} \lesssim \|f\|_Y$.

Theorem 3.3.64. *There is a constant $\epsilon_0 = \epsilon_0(m) > 0$ such that, if $|||f_0|||_X \le \epsilon_0$ then there exists a global weak solution f to* (3.53) *that satisfies, for some constant $C > 0$,*

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \le C\epsilon_0.$$

Moreover, if $F_0 = \mu + f_0 \ge 0$ then $F(t) = \mu + f(t) \ge 0$.

Proof. For any integer $n \ge 1$ we define the iterative scheme

$$\begin{cases} \partial_t f^n = \Lambda f^n + Q(f^{n-1}, f^n) \\ f^n_{|t=0} = f_0 \end{cases} \quad \forall\, n \ge 1, \quad \text{and} \quad \begin{cases} \partial_t f^0 = \Lambda f^0 \\ f^0_{|t=0} = f_0 \end{cases}.$$

Firstly, the functions f^n are well defined on X for all $t \ge 0$ thanks to the semigroup theory in Theorem 3.2.45 and the stability estimates proven below.

Step 1. Stability of the scheme. We prove by induction that

$$\forall\, n \ge 0, \forall\, t \ge 0, \quad A_n(t) := |||f^n_t|||^2_X + K\int_0^t \|f^n_\tau\|^2_Y \, d\tau \le 2\epsilon_0^2, \tag{3.54}$$

if $\epsilon_0 > 0$ is small enough. The case $n = 0$ easily follows from Proposition 3.3.58. Assume that (3.54) holds for some $n \in \mathbb{N}$. Arguing as in Proposition 3.3.63 we obtain

$$\begin{aligned} \frac{d}{dt}|||f^{n+1}|||^2_X + K\|f^{n+1}\|^2_Y &\le C|||f^n|||_X \,\|f^{n+1}\|^2_Y + C\|f^n\|_Y \,|||f^{n+1}|||^2_X \\ &\le C|||f^n|||_X \,\|f^{n+1}\|^2_Y + C\|f^n\|_Y \,|||f^{n+1}|||_X \,\|f^{n+1}\|_Y. \end{aligned}$$

Integrating from 0 to t it follows

$$\begin{aligned} A_{n+1}(t) &= |||f^{n+1}_t|||^2_X + K\int_0^t \|f^{n+1}_\tau\|^2_Y \, d\tau \\ &\le |||f_0|||^2_X + C\left(\sup_{\tau\ge 0} |||f^n_\tau|||_X\right)\int_0^t \|f^{n+1}(\tau)\|^2_Y \, d\tau \\ &\qquad + C\left(\int_0^t \|f^n(\tau)\|^2_Y \, d\tau\right)^{1/2}\left(\sup_{\tau\ge 0}|||f^{n+1}_\tau|||_X\right)\left(\int_0^t \|f^{n+1}(\tau)\|^2_Y \, d\tau\right)^{1/2} \\ &\le |||f_0|||^2_X + CA_n(t)^{1/2}A_{n+1}(t) \\ &\le |||f_0|||^2_X + C\epsilon_0 A_{n+1}(t), \end{aligned}$$

from which we conclude to (3.54) for $n+1$ if $\epsilon_0 > 0$ is small enough (so that $C\epsilon_0 \le 1/2$).

Step 2. Convergence of the scheme. Now we can prove the convergence of the scheme in X_0. Denote $d^n = f^{n+1} - f^n$ that satisfies

$$\begin{cases} \partial_t d^n = \Lambda d^n + Q(f^n, d^n) + Q(d^{n-1}, f^n), \quad \forall\, n \in \mathbb{N}^*; \\ \partial_t d^0 = \Lambda d^0 + Q(f^0, f^1). \end{cases}$$

We claim that for $\epsilon_0 > 0$ small enough, for any $n \in \mathbb{N}$ it holds

$$\forall\, t \ge 0, \forall\, n \ge 0, \quad B_n(t) := \|\!|d^n_t|\!\|^2_{X_0} + K \int_0^t \|d^n_\tau\|^2_{Y_0}\, d\tau \le (C'\epsilon_0)^{2n}, \tag{3.55}$$

for some constants $C' > 0$ that does not depend on ϵ. Let us prove the claim by induction. We start with the case $n = 0$. Denote $\overline{X}_0 := \mathcal{H}^3_x L^2_v(m_0')$ (where $m_0' \lesssim m_0 \langle v \rangle^{-(1-\sigma/2)}$, see (3.52)) then we compute

$$\begin{aligned} \frac{d}{dt} \|\!|d^0|\!\|^2_{X_0} &= \eta \langle \Lambda d^0, d^0 \rangle_{X_0} + \int_0^t \langle S_\Lambda(\tau) e^{\lambda_2 \tau} \Lambda d^0, S_\Lambda(\tau) e^{\lambda_2 \tau} d^0 \rangle_{\overline{X}_0}\, d\tau \\ &\quad + \eta \langle Q(f^0, f^1), d^0 \rangle_{X_0} + \int_0^\infty \langle S_\Lambda(\tau) e^{\lambda_2 \tau} Q(f^0, f^1), S_\Lambda(\tau) e^{\lambda_2 \tau} d^0 \rangle_{\overline{X}_0}\, d\tau \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Arguing as in Proposition 3.3.63 we get

$$I_1 + I_2 \le -K \|d^0\|^2_{Y_0}$$

and also

$$I_4 \lesssim \|\!|f^0|\!\|_{X_0}\, \|f^1\|_{Y_0}\, \|d^0\|_{Y_0}.$$

Now for the term I_3 we get thanks to Lemma 3.3.62-(i)

$$I_3 \lesssim \|\!|f^0|\!\|_{X_0}\, \|f^1\|_{Z_0}\, \|d^0\|_{Y_0} \lesssim \|\!|f^0|\!\|_{X_0}\, \|f^1\|_{Y}\, \|d^0\|_{Y_0}.$$

Gathering previous estimates yields, for any $t \ge 0$,

$$\begin{aligned} \|\!|d^0_t|\!\|^2_{X_0} + K \int_0^t \|d^0_\tau\|^2_{Y_0}\, d\tau &\le C \int_0^t \|\!|f^0_\tau|\!\|_{X_0} \left\{ \|f^1_\tau\|_{Y_0} + \|f^1_\tau\|_{Y} \right\} \|d^0_\tau\|_{Y_0}\, d\tau. \\ &\le C \left(\sup_{\tau \ge 0} \|\!|f^0_\tau|\!\|_{X_0} \right) \left(\int_0^t \|f^1_\tau\|^2_{Y}\, d\tau \right)^{1/2} \left(\int_0^t \|d^0_\tau\|^2_{Y_0}\, d\tau \right)^{1/2}, \end{aligned}$$

therefore we get

$$B_0(t) \le C \epsilon_0^2 B_0(t)^{1/2} \quad \Rightarrow \quad B(t) \le C \epsilon_0^4,$$

where we have used (3.54) for f^0 and f^1, which concludes the proof of (3.55) for $n = 0$. Assume now that (3.55) holds for some $n \in \mathbb{N}$ and let us prove (3.55) for $n+1$. We compute

$$\begin{aligned}\frac{d}{dt}|||d^{n+1}|||^2_{X_0} &= \eta\langle \Lambda d^{n+1}, d^{n+1}\rangle_{X_0} + \int_0^t \langle S_\Lambda(\tau)e^{\lambda_2\tau}\Lambda d^{n+1}, S_\Lambda(\tau)e^{\lambda_2\tau}d^{n+1}\rangle_{\overline{X}_0}\,d\tau\\
&\quad + \eta\langle Q(f^{n+1}, d^{n+1}), d^{n+1}\rangle_{X_0}\\
&\quad + \int_0^\infty \langle S_\Lambda(\tau)e^{\lambda_2\tau}Q(f^{n+1}, d^{n+1}), S_\Lambda(\tau)e^{\lambda_2\tau}d^{n+1}\rangle_{\overline{X}_0}\,d\tau\\
&\quad + \eta\langle Q(d^n, f^n), d^{n+1}\rangle_{X_0} + \int_0^\infty \langle S_\Lambda(\tau)e^{\lambda_2\tau}Q(d^n, f^n), S_\Lambda(\tau)e^{\lambda_2\tau}d^{n+1}\rangle_{\overline{X}_0}\,d\tau\\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned}$$

Arguing as in Proposition 3.3.63 we have

$$I_1 + I_2 \le -K\|d^{n+1}\|^2_{Y_0},$$

and

$$I_3 + I_4 \lesssim |||f^{n+1}|||_{X_0}\,\|d^{n+1}\|^2_{Y_0} + \|f^{n+1}\|_{Y_0}\,|||d^{n+1}|||_{X_0}\,\|d^{n+1}\|_{Y_0}.$$

The term I_6 can be estimated as I_4 and that gives us

$$I_6 \lesssim |||d^n|||_{X_0}\,\|f^n\|_{Y_0}\,\|d^{n+1}\|_{Y_0}.$$

For the last term I_5 we get using Lemma 3.3.62-(i)

$$I_5 \lesssim |||d^n|||_{X_0}\,\|f^n\|_{Z_0}\,\|d^{n+1}\|_{Y_0} \lesssim |||d^n|||_{X_0}\,\|f^n\|_{Y}\,\|d^{n+1}\|_{Y_0}.$$

Putting together all the estimates, it follows

$$\begin{aligned}B_{n+1}(t) &\le C\int_0^t |||f^{n+1}_\tau|||_{X_0}\,\|d^{n+1}_\tau\|^2_{Y_0}\,d\tau + C\int_0^t \|f^{n+1}_\tau\|_{Y_0}\,|||d^{n+1}_\tau|||_{X_0}\,\|d^{n+1}_\tau\|_{Y_0}\,d\tau\\
&\quad + C\int_0^t |||d^n_\tau|||_{X_0}\,\|f^n_\tau\|_{Y}\,\|d^{n+1}_\tau\|_{Y_0}\,d\tau\\
&\le C\left(\sup_{\tau\ge 0}\|f^{n+1}_\tau\|_{X_0}\right)\int_0^t \|d^{n+1}_\tau\|^2_{Y_0}\,d\tau\\
&\quad + C\left(\int_0^t \|f^{n+1}_\tau\|^2_{Y_0}\,d\tau\right)^{1/2}\left(\sup_{\tau\ge 0}\|d^{n+1}_\tau\|_{X_0}\right)\left(\int_0^t \|d^{n+1}_\tau\|^2_{Y_0}\,d\tau\right)^{1/2}\\
&\quad + C\left(\sup_{\tau\ge 0}\|d^{n}_\tau\|_{X_0}\right)\left(\int_0^t \|f^{n}_\tau\|^2_{Y}\,d\tau\right)^{1/2}\left(\int_0^t \|d^{n+1}_\tau\|^2_{Y_0}\,d\tau\right)^{1/2}.\end{aligned}$$

Hence it follows

$$\begin{aligned}B_{n+1}(t) &\le C\epsilon_0 B_{n+1}(t) + C\epsilon_0 B_n(t)^{1/2}B_{n+1}(t)^{1/2}\\
&\le C\epsilon_0 B_{n+1}(t) + C\epsilon_0 (C'\epsilon_0)^n B_{n+1}(t)^{1/2},\end{aligned}$$

where we have used (3.54) for f^n and f^{n+1} and also the induction hypothesis. If $\epsilon_0 > 0$ is small enough so that $C\epsilon_0 \le 1/2$, we then get

$$B_{n+1}(t) \le C\epsilon_0 (C'\epsilon_0)^n B_{n+1}(t)^{1/2} \quad\Rightarrow\quad B_{n+1}(t) \le C^2\epsilon_0^2 (C'\epsilon_0)^{2n} \le (C'\epsilon_0)^{2(n+1)}.$$

Therefore the sequence $(f^n)_n$ is a Cauchy sequence in the space $L^\infty([0,\infty);X_0) = L^\infty([0,\infty);\mathcal{H}^3_x L^2_v(m_0))$, and its limit f satisfies (3.53). We then deduce that

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \le C\epsilon_0,$$

by passing to the limit $n \to \infty$ in (3.54). Moreover, since $F_0 = \mu + f_0 \ge 0$ we easily obtain that $F(t) = \mu + f(t) \ge 0$ (see e.g. [52]). □

We can now address the problem of uniqueness and prove an a priori estimate on the difference of two solutions.

Theorem 3.3.65. *There is a constant $\epsilon_0 = \epsilon_0(m) > 0$ such that, if $|||f_0|||_X \le \epsilon_0$ then there exists a unique global weak solution $f \in L^\infty([0,\infty);X) \cap L^2([0,\infty);Y)$ to* (3.53).

Proof. Let f be the solution constructed in Theorem 3.3.64 that satisfies

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \le C\epsilon_0.$$

Assume that there is another solution g with initial data $g_0 = f_0$ and such that

$$\|g\|_{L^\infty([0,\infty);X)} + \|g\|_{L^2([0,\infty);Y)} \le C\epsilon_0.$$

The difference $f-g$ satisfies

$$\partial_t(f-g) = \Lambda(f-g) + Q(g,f-g) + Q(f-g,f),$$

with $f_0 = g_0$. We then compute the (standard) $L^2_x L^2_v(m_0)$ norm of the difference $f-g$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f-g\|^2_{L^2_xL^2_v(m_0)} &= \langle \Lambda(f-g), f-g\rangle_{L^2_xL^2_v(m_0)} + \langle Q(g,f-g), f-g\rangle_{L^2_xL^2_v(m_0)} \\ &\quad + \langle Q(f-g,f), f-g\rangle_{L^2_xL^2_v(m_0)}.\end{aligned}$$

We write $\Lambda = \mathcal{A} + \mathcal{B}$ so that we obtain

$$\langle \Lambda(f-g), f-g\rangle_{L^2_xL^2_v(m_0)} \le -K\|f-g\|^2_{L^2_xH^1_{v,*}(m_0)} + C\|f-g\|^2_{L^2_xL^2_v(m_0)}.$$

Moreover, Lemma 3.3.61-(ii) gives

$$\begin{aligned}&\langle Q(g,f-g), f-g\rangle_{L^2_xL^2_v(m_0)} \\ &\quad \le C\|g\|_{H^2_xL^2_v(m_0)}\,\|f-g\|^2_{L^2_xH^1_{v,*}(m_0)} + C\|g\|_{H^2_xH^1_v(m_0)}\,\|f-g\|^2_{L^2_xL^2_v(m_0)},\end{aligned}$$

whence, integrating in time,

$$\begin{aligned}&\int_0^t \langle Q(g_\tau, f_\tau - g_\tau), f_\tau - g_\tau\rangle_{L^2_xL^2_v(m_0)}\,d\tau \\ &\le C\sup_{\tau\in[0,t]}\|g_\tau\|_{H^2_xL^2_v(m_0)}\int_0^t \|f_\tau - g_\tau\|^2_{L^2_xH^1_{v,*}(m_0)} \\ &\quad + C\left(\int_0^t \|g_\tau\|^2_{H^2_xH^1_v(m_0)}\right)^{1/2}\left(\sup_{\tau\in[0,t]}\|f_\tau-g_\tau\|^2_{L^2_xL^2_v(m_0)} + \int_0^t \|f_\tau-g_\tau\|^2_{L^2_xL^2_v(m_0)}\right).\end{aligned}$$

Thanks to Lemma 3.3.61-(i) it follows

$$\langle Q(f-g,f), f-g\rangle_{L^2_xL^2_v(m_0)} \le C\|f-g\|_{L^2_xL^2_v(m_0)}\,\|f\|_{H^2_xH^1_{v,**}(m_0)}\,\|f-g\|_{L^2_xH^1_{v,*}(m_0)},$$

which integrating in time gives

$$\begin{aligned}
&\int_0^t \langle Q(f_\tau-g_\tau,f_\tau), f_\tau-g_\tau\rangle_{L^2_xL^2_v(m_0)}\,d\tau\\
&\le C(\sup_{\tau\in[0,t]}\|f_\tau-g_\tau\|_{L^2_xL^2_v(m_0)})\int_0^t \|f_\tau\|_{H^2_xH^1_{v,**}(m_0)}\,\|f_\tau-g_\tau\|_{L^2_xH^1_{v,*}(m_0)}\\
&\le C\left(\int_0^t \|f_\tau\|^2_{H^2_xH^1_{v,**}(m_0)}\right)^{1/2}\left(\sup_{\tau\in[0,t]}\|f_\tau-g_\tau\|^2_{L^2_xL^2_v(m_0)}+\int_0^t\|f_\tau-g_\tau\|^2_{L^2_xH^1_{v,*}(m_0)}\right),
\end{aligned}$$

and observe that $\|f\|_{L^2_t(H^2_xH^1_{v,**}(m_0))}\lesssim \|f\|_{L^2_t(Y)}\le C\epsilon_0$. Therefore

$$\begin{aligned}
&\|f_t-g_t\|^2_{L^2_xL^2_v(m_0)}+\lambda\int_0^t\|f_\tau-g_\tau\|_{L^2_xH^1_{v,*}(m_0)}\,d\tau\\
&\quad\le C\int_0^t\|f_\tau-g_\tau\|^2_{L^2_xL^2_v(m_0)}\,d\tau + C\epsilon_0\int_0^t\|f_\tau-g_\tau\|^2_{L^2_xH^1_{v,*}(m_0)}\,d\tau\\
&\qquad+C\epsilon_0\left(\sup_{\tau\in[0,t]}\|f_\tau-g_\tau\|^2_{L^2_xL^2_v(m_0)}+\int_0^t\|f_\tau-g_\tau\|^2_{L^2_xH^1_{v,*}(m_0)}\right),
\end{aligned}$$

and when $\epsilon_0>0$ is small enough we conclude the proof of uniqueness by Gronwall's inequality. □

Convergence to equilibrium in the close-to-equilibrium setting

Theorem 3.3.66. *Consider* **(H0)** *and some weight m. There is a positive constant $\epsilon_1\le\epsilon_0$ so that, if $|||f_0|||_X\le\epsilon_1$, then the unique global weak solution f to* (3.53) *(contructed in Theorems 3.3.64 and 3.3.65) verifies an exponential decay: for any $\lambda_2<\lambda_1$ there exists $C>0$ such that*

$$\forall\, t\ge 0,\quad \|f_t\|_X\le C\,e^{-\lambda_2 t}\,\|f_0\|_X,$$

where we recall that $\lambda_1>0$ is the optimal rate given by the semigroup decay in Theorem 3.2.45.

Proof. From Theorem 3.3.64 we have

$$\sup_{t\ge0}|||f(t)|||^2_X+\int_0^t\|f(\tau)\|^2_Y\,d\tau\le C\epsilon_1^2.$$

Using Proposition 3.3.63 we get, if $\epsilon_1>0$ is small enough so that $-K+C\epsilon_1\le -K/2$, and for any $\lambda_2<\lambda_1$,

$$\begin{aligned}
\frac12\frac{d}{dt}|||f|||^2_X &\le -\lambda_2|||f|||^2_X-(K-C\epsilon_1)\|f\|^2_Y\\
&\le -\lambda_2|||f|||^2_X-\frac{K}{2}\|f\|^2_Y,
\end{aligned}$$

and then we deduce an exponential convergence

$$\forall\, t \geq 0, \qquad |||f(t)|||_X \leq e^{-\lambda_2 t}\, |||f_0|||_X,$$

which implies

$$\forall\, t \geq 0, \qquad \|f(t)\|_X \leq C e^{-\lambda_2 t}\, \|f_0\|_X.$$

□

Deuxième partie

Equation de Boltzmann avec collisions inélastiques

Chapitre 4

Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting

Résumé. Nous considérons ici l'équation de Boltzmann non homogène en espace, inélastique avec un terme diffusif dans plusieurs cas : le coefficient de restitution peut être soit constant soit dépendre de la vitesse d'impact (ce qui inclut en particulier le cas des sphères dures viscoélastiques). Quand le paramètre de diffusion est suffisamment petit, nous prouvons un résultat d'existence globale de solutions dans un régime proche de l'équilibre et aussi dans un régime faiblement homogène pour le cas d'un coefficient d'inélasticité constant. C'est le tout premier résultat d'existence de solution dans une régime de collisions inélastiques (c'est-à-dire en ne prenant pas en compte le résultat d'existence de [4] établi dans un régime proche du vide). Nous étudions également le comportement en temps grand de ces solutions et prouvons une convergence exponentielle vers l'équilibre. La base de la preuve est l'étude de l'équation linéarisée pour laquelle nous prouvons de nouvelles estimations spectrales grâce à un argument perturbatif autour de l'équation de Boltzmann pour des sphères dures élastiques non homogène en espace.

Abstract. In this part, we consider the spatially inhomogeneous diffusively driven inelastic Boltzmann equation in different cases: the restitution coefficient can be constant or can depend on the impact velocity, including in particular the case of viscoelastic hard-spheres. In the weak thermalization regime, i.e when the diffusion parameter is sufficiently small, we prove existence of global solutions considering the close-to-equilibrium regime and the weakly inhomogeneous regime only in the case of a constant restitution coefficient. It is the very first existence theorem of global solution in an inelastic "collision regime" (that is excluding [4] where an existence theorem is proven in a near to the vacuum regime). We also study the long-time behavior of these solutions and prove a convergence to equilibrium with an exponential rate. The basis of the proof is the study of the linearized equation for which we prove new spectral gap estimates developing a perturbative argument around the spatially inhomogeneous equation for elastic hard spheres.

4.1 Introduction

4.1.1 Model and main result

We investigate in the present paper the Cauchy theory associated to the spatially inhomogeneous diffusively driven inelastic Boltzmann equation for hard spheres interactions and constant or non-constant restitution coefficient. More precisely, we consider hard-spheres particles described by their distribution density $f = f(t,x,v) \geq 0$ undergoing inelastic collisions in the torus in dimension $d = 3$. The spatial coordinates are $x \in \mathbb{T}^3$ (3-dimensional flat torus) and the velocities are $v \in \mathbb{R}^3$. The distribution f satisfies the following equation:

$$\partial_t f = Q_{e_\lambda}(f,f) + \lambda^\gamma \, \Delta_v f - v \cdot \nabla_x f. \tag{4.1}$$

Let us point out that in the case of a constant restitution coefficient, $e_\lambda(\cdot)$ is constant equal to $1-\lambda$ and γ is equal to 1, the equation hence becomes:

$$\partial_t f = Q_{1-\lambda}(f,f) + \lambda \, \Delta_v f - v \cdot \nabla_x f. \tag{4.2}$$

The term $\lambda^\gamma \, \Delta_v f$ represents a constant heat bath which models particles uncorrelated random accelerations between collisions. The quadratic collision operator Q_{e_λ} models the interactions of hard-spheres by inelastic binary collisions where the inelasticity is characterized by the so-called normal restitution coefficient $e_\lambda(\cdot)$ which can be, in contrast with previous contributions on the subject, constant or non-constant. In the non-constant case, this restitution coefficient quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact velocity. Namely, if v and v_* denote the velocities of two particles before collision, their respective velocities v' and v'_* after collision are such that

$$(u' \cdot \widehat{n}) = -(u \cdot \widehat{n}) \, e_\lambda(u \cdot \widehat{n}), \tag{4.3}$$

where $e_\lambda(\cdot) := e(\lambda \ \cdot)$ and $e := e(|u \cdot \widehat{n}|)$ is such that $0 \leq e \leq 1$. The unitary vector $\widehat{n} \in \mathbb{S}^2$ determines the impact direction, that is, $\widehat{n}$ stands for the unit vector that points from the v-particle center to the v_*-particle center at the instant of impact. Here above,

$$u = v - v_*, \quad u' = v' - v'_*,$$

denote respectively the relative velocity before and after collision. Assuming the granular particles to be perfectly smooth hard-spheres of mass $m = 1$, the velocities after collision v' and v'_* are given, in virtue of (4.3) and the conservation of momentum, by

$$v' = v - \frac{1+e_\lambda}{2} \, (u \cdot \widehat{n}) \, \widehat{n}, \quad v'_* = v_* + \frac{1+e_\lambda}{2} \, (u \cdot \widehat{n}) \, \widehat{n}. \tag{4.4}$$

The main assumption on $e(\cdot)$ we shall need is listed in the following (see [4] for more details).

Assumptions 4.1.67.

1. *The mapping* $r \to e(r)$ *from* $\mathbb{R}^+$ *to* $(0,1]$ *is absolutely continuous and non-increasing.*
2. *The mapping* $r \to r\,e(r)$ *is strictly increasing on* $\mathbb{R}^+$.
3. *There exist* $a, b > 0$ *and* $\overline{\gamma} > \gamma > 0$ *such that*

$$\forall\, r \geq 0, \quad |e(r) - 1 + a\, r^{\gamma}| \leq b\, r^{\overline{\gamma}}.$$

The assumptions (1) and (2) are trivially satisfied in the constant case which is enough to apply most of the results from [8]. The assumption (3) is crucial to do a fine study of spectrum of the linearized operator close to 0 in the non-constant case (see step 4 of proof of Theorem 4.2.82). Let us also emphasize that the three assumptions are met by the visco-elastic hard-spheres model which is the most physically relevant model for applications (see [20] and Subsection 4.1.2). In the remaining part of the paper, we suppose that the restitution coefficient $e(\cdot)$ is constant or satisfies Assumptions 4.1.67.

We here explain why studying such a rescaled equation is relevant in the case of weak thermalization regime. The associated stationary equation before rescaling is given by

$$Q_e(f,f) + \mu\, \Delta_v f - v \cdot \nabla_x f = 0 \tag{4.5}$$

for some positive thermalization coefficient $\mu > 0$. We then introduce the rescaled distribution $g_\lambda(x,v) := \lambda^3 f(x, \lambda v)$ if f is a solution of (4.5) of mass ρ. Using the following equalities which hold for any $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$,

$$\begin{aligned}
\lambda^2\, Q_e(f,f)(x,\lambda v) &= Q_{e_\lambda}(g_\lambda, g_\lambda)(x,v),\\
\lambda^5\, (\Delta_v f)(x,\lambda v) &= \Delta_v g_\lambda(x,v),\\
\lambda^3\, (v \cdot \nabla_x f)(x,\lambda v) &= v \cdot \nabla_x g_\lambda(x,v),
\end{aligned}$$

we obtain that g_λ satisfies

$$Q_{e_\lambda}(g_\lambda, g_\lambda) + \frac{\mu}{\lambda^3}\, \Delta_v g_\lambda - v \cdot \nabla_x g_\lambda = 0. \tag{4.6}$$

Let us notice that this scaling preserves mass and momentum and moreover, $e_\lambda(r)$ tends to 1 as λ goes to 0, the elastic restitution coefficient. We expect that formally, as λ goes to 0,

$$Q_{e_\lambda}(f,f) \simeq Q_1(f,f)$$

and thus that, as λ goes to 0, the dissipation of energy vanishes. We see that if $\mu > 0$ is fixed, then the second term of (4.6) becomes infinite in the limit $\lambda \to 0$. We thus have to choose $\mu := \mu_\lambda$ such that $\mu_\lambda\, \lambda^{-3}$ tends to 0 as λ goes to 0. Such as in [8], we can compute a parameter μ_λ such that the energy

$$\mathcal{E}_\lambda := \frac{1}{\rho} \int_{\mathbb{T}^3 \times \mathbb{R}^3} g_\lambda(x,v)\, |v|^2\, dx\, dv$$

is kept of order one in the limit $\lambda \to 0$, which gives $\mu = \mu_\lambda = \lambda^{3+\gamma}$. Equation (4.6) hence becomes

$$Q_{e_\lambda}(g_\lambda, g_\lambda) + \lambda^\gamma \, \Delta_v g_\lambda - v \cdot \nabla_x g_\lambda = 0.$$

This explains why we study the evolution equation (4.1).

In the sequel, it shall be more convenient to deal with a second, and equivalent, parametrization of the post-collisional velocities. Fix v and v_* with $v \neq v_*$ and let $\widehat{u} = u/|u|$. Performing in (4.4) the change of unknown $\sigma = \widehat{u} - 2\,(\widehat{u} \cdot \widehat{n})\,\widehat{n} \in \mathbb{S}^2$ provides an alternative parametrization of the unit sphere $\mathbb{S}^2$. In this case, the impact velocity reads $|u \cdot \widehat{n}| = |u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}$ and the post-collisional velocities v' and v'_* are then given by

$$v' = v - \frac{1+e_\lambda}{2}\,\frac{u - |u|\;\sigma}{2}, \qquad v'_* = v_* + \frac{1+e_\lambda}{2}\,\frac{u - |u|\;\sigma}{2}. \tag{4.7}$$

This representation allows us to give a precise definition of the Boltzmann collision operator in weak form by

$$\int_{\mathbb{R}^3} Q_{e_\lambda}(g,f)\,\psi\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)\,\left[\psi(v') \;-\; \psi(v)\right]|v - v_*|\,d\sigma\,dv_*\,dv, \tag{4.8}$$

for any $\psi = \psi(v)$ a suitable regular test function. Here, the post-collisional velocities v' and v'_* are defined by (4.7). Notice that

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -|u|^2\,\frac{1-\widehat{u}\cdot\sigma}{4}\,\left(1 - e_\lambda\left(|u|\,\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)^2\right). \tag{4.9}$$

The operator Q_{e_λ} defined by (4.8) preserves mass and momentum, and since the Laplacian also does so, the equation preserves mass and momentum. However, energy is not preserved either by the collisional operator (which tends to cool down the gas because of (4.9)) or by the diffusive operator (which warms it up).

The formula (4.8) suggests the natural splitting $Q_{e_\lambda} = Q^+_{e_\lambda} - Q^-_{e_\lambda}$ between gain and loss parts. The loss part $Q^-_{e_\lambda}$ can easily be defined in strong form noticing that

$$\langle Q^-_{e_\lambda}(g,f),\psi\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)\psi(v)|v - v_*|\,d\sigma\,dv_*\,dv =: \langle f L(g),\psi\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product in L^2 and L is the convolution operator

$$L(g)(v) = 4\pi(|\cdot| * g)(v).$$

In particular, we can notice that L and $Q^-_{e_\lambda}$ are independent of the normal restitution coefficient.

We also define the symmetrized (or polar form of the) bilinear collision operator $\widetilde{Q}_{e_\lambda}$ by setting

$$\begin{aligned}\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(g,h)\psi\,dv = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)h(v)|v - v_*|\,\left[\psi(v') \;+\; \psi(v'_*)\right]\,d\sigma\,dv_*\,dv \\ -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)h(v)|v - v_*|\,\left[\psi(v) \;+\; \psi(v_*)\right]\,d\sigma\,dv_*\,dv.\end{aligned} \tag{4.10}$$

In other words, $\widetilde{Q}_{e_\lambda}(g,h) = (Q_{e_\lambda}(g,h) + Q_{e_\lambda}(h,g))/2$. The formula (4.10) also suggests a splitting $\widetilde{Q}_{e_\lambda} = \widetilde{Q}^+_{e_\lambda} - \widetilde{Q}^-_{e_\lambda}$ between gain and loss parts. We can notice that we have $\widetilde{Q}^+_{e_\lambda}(g,h) = (Q^+_{e_\lambda}(g,h) + Q^+_{e_\lambda}(h,g))/2$ and $\widetilde{Q}^-_{e_\lambda}(g,h) = (Q^-_{e_\lambda}(g,h) + Q^-_{e_\lambda}(h,g))/2$.

In the elastic case ($\lambda = 0$), we can easily define the collision operator in strong form using the pre-post collisional change of variables:

$$Q_1(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left[f(v')g(v'_*) - f(v)g(v_*)\right] |v - v_*| \, dv_* \, d\sigma.$$

Our main result is the proof of existence of solutions for the non-linear problem (4.1) as well as stability and relaxation to equilibrium for these solutions. This work stands out from others because it is the first time that an existence result is obtained in the spatially inhomogeneous case in an inelastic "collision regime", in both cases of constant and non-constant coefficient of inelasticity.

We know from [8] that there exists $G_\lambda = G_\lambda(v)$ a space homogeneous solution of the stationary equation

$$Q_{e_\lambda}(f,f) + \lambda^\gamma \Delta_v f = 0$$

with mass 1 and vanishing momentum. Moreover, G_λ is unique for λ close enough to 0. We refer to Subsection 4.2.2 for more details.

Here is the main result that we obtain, a precise statement is given in Subsection 4.3.3 (Theorems 4.3.92 and 4.3.93).

Theorem 4.1.68. *Consider the functional space* $\mathcal{E}_0 = W^{s,1}_x W^{2,1}_v \left(\langle v\rangle e^{b\langle v\rangle^\beta}\right)$ *where* $b > 0$, $\beta \in (0,1)$ *and* $s > 6$. *For* λ *small enough, and for an initial datum* $f_{in} \in \mathcal{E}_0$ *close enough to the equilibrium* G_λ*, there exists a unique global solution* $f \in L^\infty_t(\mathcal{E}_0)$ *to (4.1) which furthermore satisfies*

$$\forall\, t \geq 0, \quad \|f_t - G_\lambda\|_{\mathcal{E}_0} \leq C\, e^{-\widetilde{\alpha}\, t}\, \|f_{in} - G_\lambda\|_{\mathcal{E}_0}$$

for some constructive constants C *and* $\widetilde{\alpha} > 0$.

Moreover, in the case of a constant restitution coefficient, the conclusion of the theorem also holds true taking an initial datum $f_{in} \in \mathcal{E}_0$ *close enough to a spatially homogeneous distribution* $g_{in} = g_{in}(v)$.

4.1.2 Physical and mathematical motivation

For a detailed physical introduction to granular gases we refer to [20, 30]. As can be seen from the references included in the latter, granular flows have become a subject of physical research on their own in the last decades, and for certain regimes of dilute and rapid flows, these studies are based on kinetic theory. By contrast, the mathematical kinetic theory of granular gas is rather young and began in the late 1990 decade. We refer to [68, 65] for some (short) mathematical introduction to this theory and a (non exhaustive) list of references. As explained in these papers, granular gases are composed of grains of macroscopic size with contact collisional interactions, when one does not consider other additional possible self-interaction mechanisms such as gravitation – for cosmic clouds for instance – or electromagnetism – for "dusty plasmas" for instance –. Therefore the natural assumption about the binary interaction between grains is that of inelastic hard spheres, with no loss of "tangential relative velocity" (according to the impact direction) and a loss in "normal relative velocity". This loss is quantified in some (normal) restitution coefficient. The latter is either assumed to be constant as a first approximation or can be more intricate: for instance it is a function of the modulus $|v' - v|$ of the normal relative velocity in the case of "visco-elastic hard spheres" (see [6], [7], [8] and [20]). In this paper, we consider both constant and non-constant restitution coefficients.

We restrict to the case of a small diffusion parameter (weak thermalization regime), which corresponds to small inelasticity. There are several motivations from mathematics and physics for such a choice:

– the first reason is related to the regime of validity of kinetic theory: as explained in [20, Chapter 6] for instance, the more inelasticity, the more correlations between grains are created during the binary collisions, and therefore the molecular chaos assumption, which is the core of the validity of Boltzmann's theory, suggests weak inelasticity to be the most effective;
– second, as emphasized in [20] again, the case of small inelasticity has been widely considered in physics or mathematical physics since it allows to use expansions around the elastic case, and since conversely it is an interesting question to understand the connection of the inelastic case (dissipative at the microscopic level) to the elastic case ("Hamiltonian" at the microscopic level);
– finally, this case of a small inelasticity is reasonable from the viewpoint of applications, since it applies to interstellar dust clouds in astrophysics, or sands and dusts in earth-bound experiments, and more generally to visco-elastic hard spheres whose restitution coefficient is not constant but close to 1 on the average.

Let us now describe the most physically relevant model, the one corresponding to viscoelastic hard spheres for which the restitution coefficient has been derived in [83]. For this peculiar model, $e(\cdot)$ admits the following representation as an infinite expansion series

$$e(|u \cdot \widehat{n}|) = 1 + \sum_{k=1}^{\infty} (-1)^k \, a_k |u \cdot \widehat{n}|^{k/5}, \quad u \in \mathbb{R}^3, \quad \widehat{n} \in \mathbb{S}^2 \tag{4.11}$$

where $a_k > 0$ for any $k \in \mathbb{N}$ are parameters depending on the material viscosity. We can see that in this case, $e(\cdot)$ satisfies Assumptions 4.1.67. More precisely, the assumption (3) is satisfied with $\gamma = 1/5$ and $\overline{\gamma} = 2/5$. In the case of a non-constant restitution coefficient, this is the principal example of application of the results in the paper, though, as we shall see, our results will cover more general cases.

4.1.3 Function spaces

For some given Borel weight function $m > 0$ on $\mathbb{R}^3$, let us define $L^q_v L^p_x(m)$ with $1 \le p, q \le +\infty$, as the Lebesgue space associated to the norm

$$\|h\|_{L^q_v L^p_x(m)} = \| \|h(\cdot, v)\|_{L^p_x} m(v)\|_{L^q_v}.$$

We also consider the standard higher-order Sobolev generalizations $W^{\sigma,q}_v W^{s,p}_x(m)$ for $\sigma, s \in \mathbb{N}$ defined by the norm

$$\|h\|_{W^{\sigma,q}_v W^{s,p}_x(m)} = \sum_{0 \le s' \le s,\, 0 \le \sigma' \le \sigma,\, s'+\sigma' \le \max(s,\sigma)} \| \|\nabla^{s'}_x \nabla^{\sigma'}_v h(\cdot, v)\|_{L^p_x} m(v)\|_{L^q_v}.$$

This definition reduces to the usual weighted Sobolev space $W^{s,p}_{x,v}(m)$ when $q = p$ and $\sigma = s$, and we recall the shorthand notation $H^s = W^{s,2}$.

4.1.4 Known results

Let us briefly review the existing results concerning inelastic hard spheres Boltzmann models. We shall mention that most of them are established in an homogeneous framework and that the major part of the investigation has been devoted to the particular case of a constant restitution coefficient.

For the inhomogeneous inelastic Boltzmann equation, the literature is more scarce; in this respect we mention the work [4] that treats the Cauchy problem in the case of near-vacuum data. It is worthwhile mentioning that the scarcity of results regarding existence of solutions for the inhomogeneous case is explained by the lack of entropy estimates for the inelastic Boltzmann equation; thus, well-known theories like the DiPerna-Lions renormalized solutions are no longer available. Let us now give an overview of papers dealing with homogeneous equations.

We begin by papers considering constant restitution coefficient and dealing with existence, uniqueness or properties of self-similar profiles (resp. stationary solutions) for freely cooling (resp. driven by a thermal bath) inelastic hard spheres. In the paper [18], existence of self-similar profiles or stationary solutions is assumed and *a priori* polynomial and exponential moments bounds are shown. The paper [44] completes the previous one showing existence of stationary solutions for inelastic hard spheres driven by a thermal bath, and improving the estimates on their tails of [18] into pointwise ones. The paper [65] shows, for freely cooling inelastic hard spheres, existence of self-similar profile(s) as well as propagation of regularity and damping of singularities with time. The paper [66] proves uniqueness of the stationary solution in the physical regime of a small inelasticity and provides various results on the linear stability and nonlinear stability of this stationary solution. Finally, the paper [67] gives similar answers as in [66] adding a thermal bath term. We can also mention the paper [81] which investigates the long-time behavior of the solutions for an “anomalous" gas. Existence and uniqueness of blow up profiles for this model are studied, together with the trend to equilibrium and the cooling law associated.

Let us now mention the papers dealing with inelastic hard spheres models with more general restitution coefficient. The paper [68] provides a Cauchy theory for freely cooling inelastic hard spheres with a broad family of collision kernels (including in particular restitution coefficients possibly depending on the relative velocity and/or the temperature), and studies whether the gas cools down in finite time or asymptotically, depending on the collision kernel. The paper [6] shows the generalized Haff's law yielding the optimal algebraic cooling rate of the temperature of a granular gas described by the homogeneous Boltzmann equation for inelastic interactions with non constant restitution coefficient. The paper [7] improves the previous one giving two simpler proofs of the Haff's law. The paper [8] studies uniqueness and regularity of the steady states of the diffusively driven Boltzmann equation in the physically relevant case where the restitution coefficient depends on the impact velocity including, in particular, the case of viscoelastic hard-spheres.

Our results are established in an inhomogeneous setting in a small inelasticity regime (close to the elastic one). To obtain them, we use results on the linearized elastic equation. We hence mention the results already obtained on the linearized elastic equation that we use. We denote $\mu := G_0$ the elastic equilibrium which is a Maxwellian distribution.

Let us underline the fact that most of the results on the linearized elastic operator have been obtained in spaces with a Maxwellian weight prescribed by the equilibrium (see [54, 55, 22, 48, 49, 15] for the homogeneous case and [91, 76] for the inhomogeneous one). Some improvements have been made to weights later on. For the spatially homogeneous case, in [10] a first extension of the decay estimate to L^1 with polynomial weight was obtained by an intricate nonconstructive approach based on decomposition of the solution and some dyadic decomposition of the velocity variable. This argument was then extended to L^p spaces in [97, 98]. In [75], another improvement was made, a spectral gap estimate on the space homogeneous semigroup was extended to the space $L^1_v(m)$ for a stretched exponential weight m, by constructive means, with optimal rate. We also mention that in [11], some non-constructive decay estimates were obtained in a Sobolev space in position combined with a polynomially weighted L^∞ space in velocity. Finally, the theory of enlargement of spectral gap developed in [51] gives explicit spectral gap estimates on the semigroup associated to the linearized non homogeneous operator $\mathcal{L}_0$ in $W^{s,p}_x W^{\sigma,q}_v(m)$ with polynomial or stretched exponential weight m.

4.1.5 Method of proof

The main outcome of this paper is a new Cauchy theory for the non-homogeneous Boltzmann equation for inelastic hard spheres (4.1). We prove existence and stability of solutions for this equation. In order to do so, we first establish the asymptotic stability of the linearized equation by a perturbation argument which uses the spectral analysis of the linearized elastic Boltzmann equation.

Let us explain in more details how we deal with the linearized problem, our method is in the spirit of the one in [67]. However, our study largely improves the one done in [67] in three aspects:

– we are able to deal with the spatial dependency in the torus;
– we are able to deal with non-constant restitution coefficients;
– we are able to obtain a decay estimate on the semigroup using the localization of the spectrum.

The perturbative argument around the elastic operator allows us to obtain results on the localization of the spectrum of the inelastic operator. It is based on the following facts:

– the inelastic operator can be written as the sum of a regularizing part and a dissipative part (these operators are defined through an appropriate mollification-truncation process, described later on);
– the inelastic operator is a small perturbation of the elastic one for a diffusion parameter sufficiently small;
– we know that the spectrum of the elastic operator is well localized.

To prove the first two points, we get estimates on the difference between the elastic and the inelastic collision operators which is small when taking λ close enough to 0. We establish these estimates in an inhomogeneous setting; this kind of estimates was only known to hold in an homogeneous setting (see [66] for the case of a constant restitution coefficient and [8] for the non-constant case).

About the third point, let us emphasize that equilibriums in the inelastic case do not decrease enough to belong to spaces with Maxwellian weights. Therefore, a perturbative theory close to the elastic equation is not possible in spaces of this type. But the results obtained in [51] via the theory of enlargement of spectral gap allows us to apply a perturbative theory. Indeed, estimates on the elastic collision operator are proved in spaces of type $W^{s,p}_x W^{\sigma,q}_v(m)$ where m is a polynomial or stretched exponential weight.

Using these facts, we prove our main result on the linearized inelastic operator. Its spectrum $\Sigma(\mathcal{L}_\lambda)$ is well localized: there is a constructive constant $\alpha > 0$ such that

$$\Sigma(\mathcal{L}_\lambda) \cap \{z \in \mathbb{C},\, \Re e\, z > -\alpha\} = \{\mu_\lambda, 0\},$$

0 is a four-dimensional eigenvalue (due to the conservation of mass and momentum) and $\mu_\lambda \in \mathbb{R}$, the “energy" eigenvalue, is a one-dimensional eigenvalue. We also obtain an estimate on μ_λ which is negative for λ close enough to 0. The behavior of μ_λ is linked with the fact that the energy is not preserved by the operator. Let us finally emphasize that we prove that these spectral properties imply the decay of the semigroup associated with an exponential rate.

Let us now explain how we go back to the nonlinear problem. We construct perturbative solutions close to the equilibrium or close to the spatially homogeneous case. To do so, we use the two following points:

– we introduce a dissipative Banach norm for the fully linearized operator which provides the key a priori estimate to get the “linearization trap";
– we prove bilinear estimates to control the nonlinear remainder in the equation.

As far as the close-to-equilibirum regime is concerned, the idea of the proof is to gather these two points; we can then prove that taking a sufficiently small initial data, the solution is trapped close to the equilibrium.

To deal with the weakly inhomogeneous regime, we also prove a local in time stability. We can then capture a general solution around the subset of spatially homogeneous solutions and then the general solution is driven towards equilibrium thanks to the relaxation estimates known for the spatially homogeneous solutions. Finally, we use the previous case once the stability neighborhood is entered by the solution.

4.1.6 Outline of the paper

In Section 4.2, we introduce the splitting of the inelastic linearized Boltzmann operator as the sum of a regularizing part and a dissipative part. We show that our inelastic operator is a small perturbation of the elastic one. We also make a fine study of spectrum close to 0, which allows us to prove existence of a spectral gap. We then obtain a property of semigroup decay in $W^{s,1}_x W^{2,1}_v(\langle v\rangle m)$ for a stretched exponential weight m. This section ends by the introduction of a new norm which is dissipative for the full linearized operator.

In Section 4.3, we go back to the nonlinear problem. We consider first the close-to-equilibrium regime and we state our main theorem concerning the weakly inhomogeneous regime.

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4.2 Properties of the linearized operator

4.2.1 Notations and definitions

For a given real number $a \in \mathbb{R}$, we define the half complex plane

$$\Delta_a := \{z \in \mathbb{C},\, \Re e\, z > a\}.$$

For some given Banach spaces $(E, \|\cdot\|_E)$ and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, we denote by $\mathscr{B}(E,\mathcal{E})$ the space of bounded linear operators from E to $\mathcal{E}$ and we denote by $\|\cdot\|_{\mathscr{B}(E,\mathcal{E})}$ or $\|\cdot\|_{E\to\mathcal{E}}$ the associated operator norm. We write $\mathscr{B}(E) = \mathscr{B}(E,E)$ when $E = \mathcal{E}$. We denote by $\mathscr{C}(E,\mathcal{E})$ the space of closed unbounded linear operators from E to $\mathcal{E}$ with dense domain, and $\mathscr{C}(E) = \mathscr{C}(E,E)$ in the case $E = \mathcal{E}$.

For a Banach space X and $\Lambda \in \mathscr{C}(X)$ we denote by $S_\Lambda(t)$ or $e^{\Lambda t}$, $t \geq 0$, its associated semigroup when it exists, by $\mathrm{D}(\Lambda)$ its domain, by $\mathrm{N}(\Lambda)$ its null space and by $\mathrm{R}(\Lambda)$ its range. We introduce the $\mathrm{D}(\Lambda)$-norm defined as $\|f\|_{\mathrm{D}(\Lambda)} = \|f\|_X + \|\Lambda f\|_X$ for $f \in \mathrm{D}(\Lambda)$. More generally, for $k \in \mathbb{N}$, we define

$$\|f\|_{\mathrm{D}(\Lambda^k)} = \sum_{j=0}^{k} \|\Lambda^j f\|_X, \quad f \in \mathrm{D}(\Lambda^k).$$

We also denote by $\Sigma(\Lambda)$ its spectrum, so that for any z belonging to the resolvent set $\rho(\Lambda) := \mathbb{C}\backslash\Sigma(\Lambda)$, the operator $\Lambda - z$ is invertible and the resolvent operator

$$\mathcal{R}_\Lambda(z) := (\Lambda - z)^{-1}$$

is well-defined, belongs to $\mathscr{B}(X)$ and has range equal to $\mathrm{D}(\Lambda)$. We recall that $\xi \in \Sigma(\Lambda)$ is said to be an eigenvalue if $N(\Lambda - \xi) \neq \{0\}$. Moreover an eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be isolated if there exists $r > 0$ such that

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C},\ |z - \xi| \leq r\} = \{\xi\}.$$

In the case when ξ is an isolated eigenvalue we may define $\Pi_{\Lambda,\xi} \in \mathscr{B}(X)$ the associated spectral projector by

$$\Pi_{\Lambda,\xi} := -\frac{1}{2i\pi}\int_{|z-\xi|=r'} (\Lambda - z)^{-1}\, dz$$

with $0 < r' < r$. Note that this definition is independent of the value of r' as the application $\mathbb{C} \setminus \Sigma(\Lambda) \to \mathscr{B}(X)$, $z \to \mathcal{R}_\Lambda(z)$ is holomorphic. For any $\xi \in \Sigma(\Lambda)$ isolated, it is well-known (see [57, Paragraph III-6.19]) that $\Pi^2_{\Lambda,\xi} = \Pi_{\Lambda,\xi}$, so that $\Pi_{\Lambda,\xi}$ is indeed a projector, and that the "associated projected semigroup"

$$S_{\Lambda,\xi}(t) := -\frac{1}{2i\pi}\int_{|z-\xi|=r'} e^{zt}\mathcal{R}_\Lambda(z)\, dz, \quad t > 0,$$

satisfies

$$\forall\, t > 0, \quad S_{\Lambda,\xi}(t) = \Pi_{\Lambda,\xi} S_\Lambda(t) = S_\Lambda(t)\Pi_{\Lambda,\xi}.$$

When moreover the so-called "algebraic eigenspace" $\mathrm{R}(\Pi_{\Lambda,\xi})$ is finite dimensional we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$. In that case, $\mathcal{R}_\Lambda$ is a meromorphic function on a neighborhood of ξ, with non-removable finite-order pole ξ, and there exists $\alpha_0 \in \mathbb{N}^*$ such that

$$\mathrm{R}(\Pi_{\Lambda,\xi}) = N(\Lambda - \xi)^{\alpha_0} = N(\Lambda - \xi)^{\alpha} \ \text{ for any } \ \alpha \geq \alpha_0.$$

On the other hand, for any $\xi \in \mathbb{C}$ we may also define the "classical algebraic eigenspace"

$$M(\Lambda - \xi) := \lim_{\alpha \to \infty} N(\Lambda - \xi)^{\alpha}.$$

We have then $M(\Lambda - \xi) \neq \{0\}$ if $\xi \in \Sigma(\Lambda)$ is an eigenvalue and $M(\Lambda - \xi) = \mathrm{R}(\Pi_{\Lambda,\xi})$ if $\xi \in \Sigma_d(\Lambda)$.

Finally for any $a \in \mathbb{R}$ such that

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$$

where $\xi_1, \dots, \xi_k$ are distinct discrete eigenvalues, we define without ambiguity

$$\Pi_{\Lambda,a} := \Pi_{\Lambda,\xi_1} + \dots \Pi_{\Lambda,\xi_k}.$$

We shall need the following definition on the convolution of semigroup (corresponding to composition at the level of the resolvent operators). If one considers some Banach spaces X_1, X_2, X_3, for two given functions

$$S_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_2)) \quad \text{and} \quad S_2 \in L^1(\mathbb{R}_+; \mathscr{B}(X_2, X_3)),$$

the convolution $S_2 * S_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_3))$ is defined as

$$\forall\, t \geq 0, \quad (S_2 * S_1)(t) := \int_0^t S_2(s)\, S_1(t-s)\, ds.$$

When $S_1 = S_2$ and $X_1 = X_2 = X_3$, $S^{(*\ell)}$ is defined recursively by $S^{(*1)} = S$ and for any $\ell \geq 2$, $S^{(*\ell)} = S * S^{(*(\ell-1))}$.

One can immediately see that if S_i satisfies $\|S_i(t)\|_{\mathscr{B}(X_i, X_{i+1})} \leq C_i\, t^{\alpha_i}\, e^{a_i\, t}$ for any $t \geq 0$ and some $a_i \in \mathbb{R}$, $\alpha_i \in \mathbb{N}$, $C_i \in (0, \infty)$, then

$$\forall\, t \geq 0, \quad \|S_1 * S_2(t)\|_{\mathscr{B}(X_1, X_2)} \leq C_1\, C_2\, \frac{\alpha_1!\, \alpha_2!}{(\alpha_1 + \alpha_2 + 1)!}\, t^{\alpha_1 + \alpha_2 + 1}\, e^{\max(a_1, a_2)\, t}.$$

This implies that if S satisfies $\|S(t)\|_{\mathscr{B}(X)} \leq C e^{a\, t}$ for any $t \geq 0$ and some $a \in \mathbb{R}$, $C \in (0, \infty)$, then

$$\forall\, t \geq 0, \quad \|S^{(*n)}(t)\|_{\mathscr{B}(X)} \leq C^n \frac{1}{(n-1)!}\, t^{n-1}\, e^{a\, t}.$$

Let us now introduce the notion of hypodissipative operators. If one consider a Banach space $(X, \|\cdot\|_X)$ and some operator $\Lambda \in \mathscr{C}(X)$, $(\Lambda - a)$ is said to be hypodissipative on X if there exists some norm $|||\cdot|||_X$ on X equivalent to the initial norm $\|\cdot\|_X$ such that

$$\forall\, f \in \mathrm{D}(\Lambda), \quad \exists\, \phi \in F(f) \quad \text{s.t} \quad \Re e \langle \phi, (\Lambda - a) f \rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket for the duality in X and X^* and $F(f) \subset X^*$ is the dual set of f defined by

$$F(f) = F_{|||\cdot|||_X}(f) := \left\{ \phi \in X^*,\, \langle \phi, f \rangle = |||f|||_X^2 = |||\phi|||_{X^*}^2 \right\}.$$

One classically sees (cf [51]) that if X is a Banach space and Λ is the generator of a semigroup S_Λ, for given constants $a \in \mathbb{R}$, $M > 0$ the following assertions are equivalent:

(a) $\Lambda - a$ is hypodissipative;

(b) the semigroup satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \leq M\, e^{a\, t}$, $t \geq 0$;

(c) there exists some norm $|||\cdot|||$ on X equivalent to the initial norm, and more precisely satisfying

$$\forall\, f \in X, \quad \|f\| \leq |||f||| \leq M\, \|f\|,$$

such that $\rho(\Lambda) \supset]a, \infty[$ and

$$\forall\, \lambda > a,\ \forall\, f \in \mathrm{D}(\Lambda), \quad |||(\Lambda - \lambda)\, f||| \geq (\lambda - a)\, |||f|||.$$

We refer to [51, Subsection 2.3] for further details on this subject.

4.2.2 Preliminaries on the steady states

Let us first recall results about the stationary equation

$$Q_{e_\lambda}(f,f) + \lambda^\gamma \Delta_v f = 0. \tag{4.12}$$

The main references for this subsection are [67] for the constant case and [8] for the non-constant case. We introduce the following notation: we shall say that a restitution coefficient $e(\cdot)$ satisfying Assumptions 4.1.67 is belonging to the class $\mathbb{E}_m$ for some integer $m \geq 1$ if $e(\cdot) \in \mathcal{C}^m(0,\infty)$ and

$$\forall\, k = 1, \dots, m, \quad \sup_{r \geq 0} r e^{(k)}(r) < \infty,$$

where $e^{(k)}(\cdot)$ denotes the k-th order derivative of $e(\cdot)$.

Remark 4.2.69. *For the physically relevant case of visco-elastic hard-spheres, the restitution coefficient $e(\cdot)$ is given by (4.11) but admits also the following implicit representation (see [20]):*

$$\forall\, r > 0, \quad e(r) + a r^{\frac{1}{5}} e^{\frac{3}{5}}(r) = 1$$

for some $a > 0$. Then, it is possible to deduce from such representation that $e(\cdot)$ belongs to the class $\mathbb{E}_m$ for any integer $m \geq 1$.

In [8, Theorem 4.5], the authors state that if $e(\cdot)$ belongs to the class $\mathbb{E}_m$ for some integer $m \geq 4$, there exists $\lambda^\dagger \in (0,1]$ such that for any $\lambda \in [0,\lambda^\dagger)$, there exists a unique solution in L^1_2 of (4.12) of mass 1 and vanishing momentum. We denote G_λ this solution.

It is also proved in [8, Proposition 3.3] that there exist $A > 0$, $M > 0$ such that for any $\lambda \in (0,\lambda^\dagger]$, G_λ satisfies

$$\int_{\mathbb{R}^3} G_\lambda(v)\, e^{A\,|v|^{3/2}}\, dv \leq M. \tag{4.13}$$

Let us point out that in the case of a constant coefficient, these results were already established. In [18, Theorem 1] and [44, Theorem 5.2 & Lemma 7.2], existence of solutions and regularity estimates are proved. In [67, Section 2.1], it is proved that these estimates are uniform in terms of the coefficient of inelasticity and in [67, Theorem 1.2], uniqueness of steady states is proved for a sufficiently small coefficient of inelasticity.

We denote $m(v) = e^{b\langle v\rangle^\beta}$, $b > 0$ and $\beta \in (0,1)$. We now state several lemmas on steady states G_λ which are straightforward consequences of results from [67] and [8]. We shall use them several times in what follows. First, we recall a result of interpolation (see for example [66, Lemma B.1]) which is going to be very useful.

Lemma 4.2.70. *Consider k, $q \in \mathbb{N}$. Then, there exists $C > 0$ such that for any $h \in H^{k'}_v \cap L^1_v(m^{12})$ with $k' = 8k + 7(1+3/2)$*

$$\|h\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq C \|h\|^{1/8}_{H^{k'}_v} \|h\|^{1/8}_{L^1_v(m^{12})} \|h\|^{3/4}_{L^1_v(m)}.$$

Let us now prove estimate on Sobolev norm of G_λ.

Lemma 4.2.71. *Let k, $q \in \mathbb{N}$. We denote $k' = 8k + 7(1 + 3/2)$. If $e(\cdot)$ belongs to the space $\mathbb{E}_{k'+1}$, then there exists $C > 0$ such that*

$$\forall\, \lambda \in (0, \lambda^\dagger], \quad \|G_\lambda\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq C.$$

Proof. We deduce from (4.13) that there exists $C > 0$ such that for any $\lambda \in (0, \lambda^\dagger]$, $\|G_\lambda\|_{L^1_v(m)} \leq C$ and $\|G_\lambda\|_{L^1_v(m^{12})} \leq C$. We now use [8, Theorem 3.6], it gives us the following:

$$\forall\, q \in \mathbb{N},\ \forall\, \ell \in [0, k'], \qquad \sup_{\lambda \in (0,\lambda^\dagger]} \|G_\lambda\|_{H^\ell_v(\langle v\rangle^q)} < \infty.$$

Gathering the previous estimates and using Lemma 4.2.70, we obtain the result. Let us mention that in the case of a constant coefficient, we can prove this result using [67, Proposition 2.1]. □

Let us now give an estimate on the difference between G_λ and G_0, the elastic equilibrium which is a Maxwelian distribution.

Lemma 4.2.72. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$. We denote $k' = 8k + 7(1 + 3/2)$. If $e(\cdot)$ belongs to the space $\mathbb{E}_{k'+1}$, then there exists a function $\varepsilon_1(\lambda)$ such that for any $\lambda \in (0, \lambda^\dagger]$*

$$\|G_\lambda - G_0\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq \varepsilon_1(\lambda) \quad \textit{with} \quad \varepsilon_1(\lambda) \xrightarrow[\lambda \to 0]{} 0.$$

Proof. Theorem 4.1 from [8] implies that

$$\|G_\lambda - G_0\|_{H^{k'}_v} \xrightarrow[\lambda \to 0]{} 0.$$

Using this estimate with Lemma 4.2.70 and Lemma 4.2.71, it yields the result. We here mention that in the case of a constant coefficient, we can conclude using [67, Lemma 4.3]. □

4.2.3 The linearized operator and its splitting

Considering the linearization $f = G_\lambda + h$, we obtain at first order the linearized equation around the equilibrium G_λ

$$\partial_t h = \mathcal{L}_\lambda h := Q_{e_\lambda}(G_\lambda, h) + Q_{e_\lambda}(h, G_\lambda) + \lambda^\gamma\, \Delta_v h - v \cdot \nabla_x h, \tag{4.14}$$

for $h = h(t, x, v)$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$.

We define the operator $\widehat{Q}_{e_\lambda}$ by

$$\widehat{Q}_{e_\lambda}(h) = Q_{e_\lambda}(G_\lambda, h) + Q_{e_\lambda}(h, G_\lambda) = 2\, \widetilde{Q}_{e_\lambda}(h, G_\lambda),$$

where $\widetilde{Q}_{e_\lambda}$ is defined in (4.10). Using the weak formulation, we have

$$\int_{\mathbb{R}^3} \widehat{Q}_{e_\lambda}(h)\psi\, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} G_\lambda(v) h(v_*) |v - v_*| \left[\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)\right]\, d\sigma\, dv_*\, dv$$

for any test function ψ.

Decomposition of the linearized operator

Let us introduce the decomposition of the linearized operator $\mathcal{L}_\lambda$. For any $\delta \in (0,1)$, we consider $\Theta_\delta = \Theta_\delta(v, v_*, \sigma) \in \mathcal{C}^\infty$ bounded by one, which equals one on

$$\Big\{|v| \leq \delta^{-1} \text{ and } 2\delta \leq |v - v_*| \leq \delta^{-1} \text{ and } |\cos\theta| \leq 1 - 2\delta\Big\}$$

and whose support is included in

$$\Big\{|v| \leq 2\delta^{-1} \text{ and } \delta \leq |v - v_*| \leq 2\delta^{-1} \text{ and } |\cos\theta| \leq 1 - \delta\Big\}.$$

We introduce the following splitting of the linearized elastic collisional operator $\widehat{Q}_1$ defined as $\widehat{Q}_1(h) = Q_1(G_0, h) + Q_1(h, G_0)$:

$$\widehat{Q}_1 = \widehat{Q}^{+,*}_{1,S} + \widehat{Q}^{+,*}_{1,R} - L(G_0)$$

with the truncated operator

$$\widehat{Q}^{+,*}_{1,S}(h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \Theta_\delta \big[G_0(v'_*)h(v') \ + \ G_0(v')h(v'_*) - G_0(v)h(v_*)\big] \ |v - v_*| \, dv_* \, d\sigma,$$

the corresponding remainder operator

$$\widehat{Q}^{+,*}_{1,R}(h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} (1 - \Theta_\delta) \big[G_0(v'_*)h(v') \ + \ G_0(v')h(v'_*) - G_0(v)h(v_*)\big] \ |v - v_*| \, dv_* \, d\sigma$$

and

$$L(G_0) = 4\pi \ (G_0 * |\cdot|).$$

We can then write a decomposition for the full linearized operator $\mathcal{L}_\lambda$:

$$\begin{aligned} \mathcal{L}_\lambda h &= \widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h) + \widehat{Q}_1(h) + \lambda^\gamma \, \Delta_v h - v \cdot \nabla_x h \\ &= \widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h) + \widehat{Q}^{*,+}_{1,S}(h) + \widehat{Q}^{+,*}_{1,R}(h) - L(G_0)\, h + \lambda^\gamma \, \Delta_v h - v \cdot \nabla_x h. \end{aligned}$$

Let us denote

$$\mathcal{A}_\delta h := \widehat{Q}^{*,+}_{1,S}(h)$$

and

$$\mathcal{B}_{\lambda,\delta} h := \widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h) + \widehat{Q}^{+,*}_{1,R}(h) + \lambda^\gamma \, \Delta_v h - v \cdot \nabla_x h - L(G_0)\, h.$$

Thanks to the truncation, we can use the so-called Carleman representation (see [93, Chapter 1, Section 4.4]) and write the truncated operator $\mathcal{A}_\delta$ as an integral operator

$$\mathcal{A}_\delta(h)(v) = \int_{\mathbb{R}^3} k_\delta(v, v_*)\, h(v_*)\, dv_* \tag{4.15}$$

for some smooth kernel $k_\delta \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3)$.

We also introduce the collision frequency $\nu := L(G_0)$ which satisfies $\nu(v) \approx \langle v \rangle$ i.e there exist some constants ν_0, $\nu_1 > 0$ such that:

$$\forall\, v \in \mathbb{R}^3, \quad 0 < \nu_0 \leq \nu_0 \langle v \rangle \leq \nu(v) \leq \nu_1 \langle v \rangle. \tag{4.16}$$

Spaces at stake

Let us consider the three Banach spaces

$$\begin{aligned}\mathcal{E}_1 &= W^{s+2,1}_x W^{4,1}_v(\langle v\rangle^2 m),\\ \mathcal{E}_0 &= W^{s,1}_x W^{2,1}_v(\langle v\rangle m),\\ \mathcal{E}_{-1} &= W^{s-1,1}_x L^1_v(m)\end{aligned}$$

for some $s \in \mathbb{N}$ such that $s/2 > 3$ (this restriction is used in the proof of Lemma 4.3.91 to get a Sobolev embedding).

In the remaining part of the paper, we suppose that the following assumption on $e(\cdot)$ holds:

Assumption 4.2.73. *The coefficient of restitution $e(\cdot)$ belongs to $\mathbb{E}_{k^\dagger+1}$ where $k^\dagger := 32 + 7(1 + 3/2)$.*

It allows us to get uniform bounds on the $\mathcal{E}_j$-norms of G_λ and uniform estimates on the $\mathcal{E}_j$-norms of the difference $G_\lambda - G_0$ for $j = -1, 0, 1$ (thanks to Lemmas 4.2.71 and 4.2.72).

The operator $\mathcal{L}_\lambda$ is bounded from $\mathcal{E}_j$ to $\mathcal{E}_{j-1}$ for $j = 0, 1$. The operators Δ_v and $v \cdot \nabla_x$ are clearly bounded from $\mathcal{E}_j$ to $\mathcal{E}_{j-1}$. As far as $\widehat{Q}_{e_\lambda}$ is concerned, we are going to use the result of interpolation Lemma 4.2.70.

Lemma 4.2.74. *Let us consider k, $q \in \mathbb{N}$. We denote $k' = 8k+7(1+3/2)$. If $e(\cdot)$ belongs to the space $\mathbb{E}_{k'+1}$, then $\widehat{Q}_{e_\lambda}$ is bounded from $W^{s,1}_x W^{k,1}_v(\langle v\rangle^{q+1} m)$ to $W^{s,1}_x W^{k,1}_v(\langle v\rangle^{q} m)$.*

Proof. As far as the case of a constant coefficient is concerned, Proposition 3.1 from [66] gives us

$$\|\widehat{Q}_{e_\lambda}(h)\|_{L^1_v(\langle v\rangle^q m)} \le C\|G_\lambda\|_{L^1_v(\langle v\rangle^{q+1} m)}\|h\|_{L^1_v(\langle v\rangle^{q+1} m)} \le C\|h\|_{L^1_v(\langle v\rangle^{q+1} m)},$$

where the last inequality comes from Lemma 4.2.71. Concerning the case of a non-constant coefficient, we use both Lemma 4.2.71 and [5, Theorem 1] and we get:

$$\|\widehat{Q}_{e_\lambda}(h)\|_{L^1_v(\langle v\rangle^q m)} \le C\|G_\lambda\|_{L^1_v(\langle v\rangle^{q+1} m)}\|h\|_{L^1_v(\langle v\rangle^{q+1} m)} \le C\|h\|_{L^1_v(\langle v\rangle^{q+1} m)}.$$

The x-derivatives commute with the operator $\widehat{Q}_{e_\lambda}$, therefore we can do the proof with $s = 0$ without loss of generality. We first look at the case $L^1_x L^1_v(\langle v\rangle^q m)$ before treating the v-derivatives. Using Fubini theorem and the previous inequalities, we obtain

$$\|\widehat{Q}_{e_\lambda} h\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le C\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}.$$

We now treat the case $L^1_x W^{1,1}_v(\langle v\rangle^q m)$. We use the property

$$\partial_v Q^{\pm}_{e_\lambda}(f,g) = Q^{\pm}_{e_\lambda}(\partial_v f, g) + Q^{\pm}_{e_\lambda}(f, \partial_v g). \tag{4.17}$$

We then compute

$$\partial_v \widehat{Q}_{e_\lambda} h = Q_{e_\lambda}(\partial_v G_\lambda, h) + Q_{e_\lambda}(G_\lambda, \partial_v h) + Q_{e_\lambda}(\partial_v h, G_\lambda) + Q_{e_\lambda}(h, \partial_v G_\lambda).$$

Using Lemma 4.2.71, [66, Proposition 3.1] in the constant case and [5, Theorem 1] in the non-constant case, the $L^1_v(\langle v\rangle^q m)$-norm of each term can be bounded by $C\|h\|_{W^{1,1}_v(\langle v\rangle^{q+1}m)}$. Again using Fubini theorem, we deduce that

$$\|\partial_v \widehat{Q}_{e_\lambda} h\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le C\|h\|_{L^1_x W^{1,1}_v(\langle v\rangle^{q+1}m)}.$$

The higher-order terms are dealt with in a similar manner, which concludes the proof. □

Under the assumptions made on $e(\cdot)$, using the previous lemma, we can conclude that $\widehat{Q}_{e_\lambda}$ is bounded from $\mathcal{E}_j$ to $\mathcal{E}_{j-1}$ for $j=0,1$.

4.2.4 Hypodissipativity of $\mathcal{B}_{\lambda,\delta}$ and boundedness of $\mathcal{A}_\delta$

Lemma 4.2.75. *Let us consider $s \ge 0$, $k \ge 0$ and $q \ge 0$. We denote $k' = 8k+7(1+3/2)$. If $e(\cdot)$ belongs to the space $\mathbb{E}_{k'+1}$, then there exist $\lambda_0 \in (0,\lambda^\dagger)$, $\delta > 0$ and $\alpha_0 > 0$ such that for any $\lambda \in [0,\lambda_0]$, $\mathcal{B}_{\lambda,\delta} + \alpha_0$ is hypodissipative in $W^{s,1}_x W^{k,1}_v(\langle v\rangle^q m)$.*

Proof. Observe first that the x-derivatives commute with the operator $\mathcal{B}_{\lambda,\delta}$, therefore we can do the proof for $s=0$ without loss of generality.

We consider a solution h_t to the linear equation $\partial_t h_t = \mathcal{B}_{\lambda,\delta}(h_t)$ with given initial datum h_0. We first look at the case $L^1_x L^1_v(\langle v\rangle^q m)$ before treating the v-derivatives. We compute

$$\begin{aligned}
\frac{d}{dt}\|h_t\|_{L^1_x L^1_v((\langle v\rangle^q m)} =& \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |h_t|\,dx\,\langle v\rangle^q m(v)\,dv \\
=& \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \partial_t h_t \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
=& \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \mathcal{B}_{\lambda,\delta}(h_t) \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
=& \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} (\widehat{Q}_{e_\lambda}(h_t) - \widehat{Q}_1(h_t)) \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \widehat{Q}^{+,*}_{1,R}(h_t) \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
&+ \lambda^\gamma \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v h_t \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} v\cdot\nabla_x h_t \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \nu\, h_t \,\mathrm{sign}(h_t)\,dx\,\langle v\rangle^q m(v)\,dv \\
=:& I_1(h_t) + I_2(h_t) + I_3(h_t) + I_4(h_t) + I_5(h_t).
\end{aligned}$$

We first deal with I_1 splitting the difference $\widehat{Q}_{e_\lambda} - \widehat{Q}_1$ into several parts and using that $Q^-_{e_\lambda} = Q^-_1$:

$$\begin{aligned}\widehat{Q}_{e_\lambda} h - \widehat{Q}_1 h =& \, Q^+_{e_\lambda}(h, G_\lambda) - Q^+_1(h, G_\lambda) + Q^+_1(h, G_\lambda - G_0) \\ &+ Q^+_{e_\lambda}(G_\lambda, h) - Q^+_1(G_\lambda, h) + Q^+_1(G_\lambda - G_0, h) \\ &- Q^-_1(h, G_\lambda - G_0) - Q^-_1(G_\lambda - G_0, h) \\ =& \, 2\left[\widetilde{Q}^+_{e_\lambda}(h, G_\lambda) - \widetilde{Q}^+_1(h, G_\lambda) + \widetilde{Q}^+_1(h, G_\lambda - G_0) - \widetilde{Q}^-_1(h, G_\lambda - G_0)\right].\end{aligned}$$

We now use a result given by [66, Proposition 3.1] which can be easily extended to others weights of type $\langle v\rangle^q m$. We can treat together the terms $\widetilde{Q}^+_1(h, G_\lambda - G_0)$ and $\widetilde{Q}^-_1(h, G_\lambda - G_0)$. Because of [66, Proposition 3.1], their $L^1_v(\langle v\rangle^q m)$-norm are bounded from above by $C \, \|G_\lambda - G_0\|_{L^1_v(\langle v\rangle^{q+1} m)} \|h\|_{L^1_v(\langle v\rangle^{q+1} m)}$. Then, using Lemma 4.2.72, we obtain

$$\|\widetilde{Q}^\pm_1(h, G_\lambda - G_0)\|_{L^1_v(\langle v\rangle^q m)} \le C \, \varepsilon_1(\lambda) \|h\|_{L^1_v(\langle v\rangle^{q+1} m)} \tag{4.18}$$

with $\varepsilon_1(\lambda) \xrightarrow[\lambda \to 0]{} 0$. Concerning the term $\widetilde{Q}^+_{e_\lambda}(h_t, G_\lambda) - \widetilde{Q}^+_1(h_t, G_\lambda)$, we use [8, Theorem 3.11] (we can use [66, Proposition 3.2] for the constant case) and Lemma 4.2.71. It gives us that there exists $\lambda_1 \in (0, \lambda^\dagger]$ such that for any $\lambda \in (0, \lambda_1)$:

$$\begin{aligned}\|\widetilde{Q}^+_{e_\lambda}(h, G_\lambda) - \widetilde{Q}^+_1(h, G_\lambda)\|_{L^1_v(\langle v\rangle^q m)} &\le C \lambda^{\frac{\gamma}{8+3\gamma}} \, \|G_\lambda\|_{W^{1,1}_v(\langle v\rangle^{q+1} m)} \, \|h\|_{L^1_v(\langle v\rangle^{q+1} m)} \\ &\le C \varepsilon_2(\lambda) \, \|h\|_{L^1_v(\langle v\rangle^{q+1} m)}\end{aligned} \tag{4.19}$$

with $\varepsilon_2(\lambda) \xrightarrow[\lambda \to 0]{} 0$. In [8] and [66], the results are only stated in the case $q = 0$ but it is easy to extend these results using the fact that $\langle v'\rangle^q \le C \, \langle v\rangle^q \, \langle v_*\rangle^q$.

Gathering (4.18) and (4.19), we thus obtain

$$I_1(h) \le \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \left|\widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h)\right| \, dx \, \langle v\rangle^q m(v) \, dv \le \varepsilon(\lambda) \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} \tag{4.20}$$

with $\varepsilon(\lambda) \xrightarrow[\lambda \to 0]{} 0$.

As far as I_2 is concerned, we first recall that [75, Proposition 2.1] establishes that there holds

$$\forall \, h \in L^1_v(\langle v\rangle m), \quad \|\widehat{Q}^{+,*}_{1,R}(h)\|_{L^1_v(m)} \le \Lambda(\delta) \|h\|_{L^1_v(\langle v\rangle m)} \quad \text{with} \quad \Lambda(\delta) \xrightarrow[\delta \to 0]{} 0,$$

where however the definition of Θ_δ is slightly different and only the case $q = 0$ is treated. But it is straightforward to extend the proof to the present situation. We hence have

$$I_2(h) \le \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} |\widehat{Q}^{+,*}_{1,R}(h_t)| \, dx \, \langle v\rangle^q m(v) \, dv \le \Lambda(\delta) \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}, \tag{4.21}$$

with $\Lambda(\delta) \xrightarrow[\delta \to 0]{} 0$.

Concerning the term with the Laplacian, we write performing two integrations by parts

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v h_t \,\mathrm{sign}(h_t)\, \langle v\rangle^q m \, dv\, dx &= -\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |\nabla_v h|^2 \,\mathrm{sign}'(h)\, \langle v\rangle^q m\, dv\, dx \\ &\quad - \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \nabla_v h \,\mathrm{sign}(h) \cdot \nabla_v(\langle v\rangle^q m(v))\, dv\, dx \\ &\le \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \nabla_v |h| \cdot \nabla_v(\langle v\rangle^q m)\, dv\, dx \\ &= \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |h|\, \Delta_v(\langle v\rangle^q m)\, dv\, dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |h|\, \langle v\rangle^q m \, \frac{\Delta_v(\langle v\rangle^q m)}{\langle v\rangle^q m}\, dx\, dv.\end{aligned}$$

Since $\Delta_v(\langle v\rangle^q m)/(\langle v\rangle^q m)$ is bounded in $\mathbb{R}^3$, we can write

$$I_3(h) \le C\,\lambda^\gamma \,\|h\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le C\,\lambda^\gamma\, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}. \tag{4.22}$$

We notice that

$$I_4(h) = 0 \tag{4.23}$$

because the term $v \cdot \nabla_x h$ has a divergence structure in x.

Finally, let us deal with I_5. We use property (4.16), more precisely the fact that $\nu(v)$ is bounded below by $\nu_0\langle v\rangle$:

$$I_5(h) = -\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |h|\, dx\, \nu\, \langle v\rangle^q m(v)\, dv \le -\nu_0\, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}. \tag{4.24}$$

Gathering (4.20), (4.21), (4.22), (4.23) and (4.24), we obtain that for any $\lambda \in (0, \lambda_1)$

$$\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \mathcal{B}_{\lambda,\delta} h \,\mathrm{sign}(h)\, dx\, \langle v\rangle^q m(v)\, dv \le (\Lambda(\delta) + \varepsilon(\lambda) + C\lambda^\gamma - \nu_0)\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}.$$

We choose $\lambda_0 \in (0, \lambda_1]$ small enough so that for any $\lambda \in [0, \lambda_0]$, $\varepsilon(\lambda) + C\lambda^\gamma < \nu_0$. Then, we choose δ close enough to 0 in order to have

$$\alpha_0 := -\left(\Lambda(\delta) + \max_{\lambda\in[0,\lambda_0]} [\varepsilon(\lambda) + C\lambda^\gamma] - \nu_0\right) > 0. \tag{4.25}$$

We hence have

$$\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \mathcal{B}_{\lambda,\delta} h \,\mathrm{sign}(h)\, dx\, \langle v\rangle^q m(v)\, dv \le -\alpha_0\, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}.$$

In particular, we deduce that for any $\lambda \in [0, \lambda_0]$, $\mathcal{B}_{\lambda,\delta} + \alpha_0$ is dissipative in $L^1_x L^1_v(\langle v\rangle^q m)$.

Let us now treat the v-derivatives. We are going to deal with the case $L^1_x W^{1,1}_v(\langle v\rangle^q m)$, the higher-order cases are similar. Thanks to (4.17), we compute the evolution of the v-derivatives:

$$\partial_t \partial_v h_t = \partial_v \left(\widehat{Q}^{+,*}_{1,R}(h_t) - \nu h_t\right) + \partial_v\left((\widehat{Q}_{e_\lambda} - \widehat{Q}_1)(h_t)\right) + \lambda^\gamma \Delta_v \partial_v h_t - \partial_x h_t - v\cdot\nabla_x \partial_v h_t.$$

Let us treat the first term:

$$\partial_v \left(\widehat{Q}^{+,*}_{1,R}(h) - \nu h \right) = \widehat{Q}^{+,*}_{1,R}(\partial_v h) - \nu\, \partial_v h + \mathcal{R}h$$

with

$$\mathcal{R}h := Q_1(h, \partial_v G_0) + Q_1(\partial_v G_0, h) - (\partial_v \mathcal{A}_\delta)(h) + \mathcal{A}_\delta(\partial_v h).$$

Using the form (4.15) of the operator $\mathcal{A}_\delta$ and performing one integration by part, we can show that

$$\|(\partial_v \mathcal{A}_\delta)(h)\|_{L^1_x L^1_v(\langle v\rangle^q m)} + \|\mathcal{A}_\delta(\partial_v h)\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le C_\delta \, \|h\|_{L^1_x L^1_v(\langle v\rangle^q m)}.$$

Combining this inequality with estimates [67, Proposition 3.1] on the elastic bilinear operator Q_1 of, we obtain

$$\|\mathcal{R}h\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le \, C_\delta \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}$$

for some constant $C_\delta > 0$.

Let us now deal with the second term coming from the difference $\widehat{Q}_{e_\lambda} - \widehat{Q}_1$:

$$\begin{aligned}\partial_v \left((\widehat{Q}_{e_\lambda} - \widehat{Q}_1) h \right) = & \, (\widehat{Q}_{e_\lambda} - \widehat{Q}_1)(\partial_v h) \\ & + 2 \left[\widetilde{Q}^+_{e_\lambda}(h, \partial_v G_\lambda) - \widetilde{Q}^+_1(h, \partial_v G_\lambda) \right] \\ & + 2 \left[\widetilde{Q}^+_1(h, \partial_v (G_\lambda - G_0)) - \widetilde{Q}^-_1(h, \partial_v (G_\lambda - G_0)) \right].\end{aligned}$$

Arguing as before, we obtain

$$\|\widetilde{Q}^+_{e_\lambda}(h, \partial_v G_\lambda) - \widetilde{Q}^+_1(h, \partial_v G_\lambda)\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le \varepsilon(\lambda) \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}$$

and

$$\|\widetilde{Q}^+_1(h, \partial_v (G_\lambda - G_0)) - \widetilde{Q}^-_1(h, \partial_v (G_\lambda - G_0))\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le \varepsilon(\lambda) \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}$$

with $\varepsilon(\lambda) \xrightarrow[\alpha \to 0]{} 0$.

All together, we deduce that

$$\partial_t \partial_v h_t = \mathcal{B}_{\lambda,\delta} \partial_v h_t + \mathcal{R}'(h_t)$$

with

$$\|\mathcal{R}' h\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le C_\delta \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \varepsilon(\lambda) \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h\|_{L^1_x L^1_v(\langle v\rangle^q m)}.$$

We now use the proof of the previous case to finally deduce the following estimate:

$$\begin{aligned}\frac{d}{dt} \|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le & - \alpha_0 \|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + C_\delta \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} \\ & + \varepsilon(\lambda) \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h\|_{L^1_x L^1_v(\langle v\rangle^q m)},\end{aligned}$$

where α_0 is defined in (4.25).

Again using the proof of the previous case, we also have:

$$\begin{aligned}&\frac{d}{dt}\left(\|h_t\|_{L^1_xL^1_v(\langle v\rangle^q m)}+\|\nabla_x h_t\|_{L^1_xL^1_v(\langle v\rangle^q m)}\right)\\ \leq &-\alpha_0\left(\|h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\|\nabla_x h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}\right).\end{aligned}$$

We now introduce the norm

$$\|h\|_* := \|h\|_{L^1_xL^1_v(\langle v\rangle^q m)}+\|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^q m)}+\eta\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^q m)}$$

for some $\eta>0$ to be fixed later. We deduce

$$\begin{aligned}\frac{d}{dt}\|h_t\|_* &\leq -\alpha_0\left(\|h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\|\nabla_x h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\eta\|\nabla_v h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}\right)\\ &\quad+\eta\left(C_\delta\,\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\varepsilon(\lambda)\,\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}\right)\\ &\leq(-\alpha_0+o(\eta))\left(\|h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\|\nabla_x h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\eta\|\nabla_v h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}\right)\end{aligned}$$

with $o(\eta)\xrightarrow[\eta\to 0]{}0$. We choose η close enough to 0 so that $\alpha_1:=\alpha_0-o(\eta)>0$. We thus obtain

$$\begin{aligned}\frac{d}{dt}\|h_t\|_* &\leq -\alpha_1\left(\|h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\|\nabla_x h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}+\eta\|\nabla_v h_t\|_{L^1_xL^1_v(\langle v\rangle^{q+1} m)}\right)\\ &\leq -\alpha_1\|h_t\|_*,\end{aligned}$$

with $\alpha_1>0$, which concludes the proof. □

Let us clarify what implies the previous lemma giving the following result:

Lemma 4.2.76. *Under the assumptions made on $e(\cdot)$, there exist $\lambda_0\in(0,\lambda^\dagger]$, $\delta>0$ and $\alpha_0>0$ such that for any $\lambda\in[0,\lambda_0]$, $\mathcal{B}_{\lambda,\delta}+\alpha_0$ is hypodissipative in $\mathcal{E}_j$, $j=-1,0,1$.*

The boundedness of $\mathcal{A}_\delta$ is treated in [51]. Let us recall Lemma 4.16 of [51].

Lemma 4.2.77. *For any $s\in\mathbb{N}$, the operator $\mathcal{A}_\delta$ maps $L^1_v(\langle v\rangle)$ into H^s_v functions with compact support, with explicit bounds (depending on δ) on the $L^1_v(\langle v\rangle)\to H^s_v$ norm and on the size of the support.*

More precisely, there are two constants $C_{s,\delta}$ and R_δ such that for any $h\in L^1_v(\langle v\rangle)$

$$K:=\operatorname{supp}\mathcal{A}_\delta h\subset B(0,R_\delta),\quad \|\mathcal{A}_\delta h\|_{H^s_v(K)}\leq C_{s,\delta}\|h\|_{L^1_v(\langle v\rangle)}.$$

In particular, we deduce that $\mathcal{A}_\delta$ is in $\mathscr{B}(\mathcal{E}_j)$ for $j=-1,0,1$.

4.2.5 Regularization properties of $T_n := \left(\mathcal{A}_\delta\, S_{\mathcal{B}_{\lambda,\delta}}\right)^{(*n)}$

Let us consider λ_0 and α_0 provided by Lemma 4.2.76.

Lemma 4.2.78. *Let λ be in $(0, \lambda_0)$. The time indexed family T_n of operators satisfies the following: for any $\alpha_0' \in (0, \alpha_0)$, there are some constructive constants $C_\delta > 0$ and R_δ such that for any $t \geq 0$*

$$supp\, T_n(t)h \subset K := B(0, R_\delta),$$

and

$$\|T_1(t)h\|_{W^{s+1,1}_{x,v}(K)} \leq C\frac{e^{-\alpha_0' t}}{t}\|h\|_{W^{s,1}_{x,v}(\langle v\rangle m)}, \quad \textit{if } s \geq 1; \tag{4.26}$$

$$\|T_2(t)h\|_{W^{s+1/2,1}_{x,v}(K)} \leq Ce^{-\alpha_0' t}\|h\|_{W^{s,1}_{x,v}(\langle v\rangle m)}, \quad \textit{if } s \geq 0. \tag{4.27}$$

Proof. We first consider $h_0 \in W^{s,1}_{x,v}(\langle v\rangle m)$, $s \in \mathbb{N}$. Using Lemma 4.2.77 and the fact that the x-derivatives commute with both $\mathcal{A}_\delta$ and $\mathcal{B}_{\lambda,\delta}$ and thus with $T_1(t)$, we get

$$\|T_1(t)h_0\|_{W^{s,1}_x W^{s+1,1}_v(K)} = \|\mathcal{A}_\delta\, S_{\mathcal{B}_{\lambda,\delta}}(t)h_0\|_{W^{s,1}_x W^{s+1,1}_v(K)} \leq C\, \|S_{\mathcal{B}_{\lambda,\delta}}(t)h_0\|_{W^{s,1}_{x,v}(K)}.$$

We then use that $\mathcal{B}_{\lambda,\delta} + \alpha_0$ is dissipative in $W^{s,1}_{x,v}(\langle v\rangle m)$ (Lemma 4.2.76) to obtain

$$\|T_1(t)h_0\|_{W^{s,1}_x W^{s+1,1}_v(K)} \leq Ce^{-\alpha_0 t}\|h_0\|_{W^{s,1}_{x,v}(\langle v\rangle m)}. \tag{4.28}$$

Assume now $h_0 \in W^{s,1}_x W^{s+1,1}_v(\langle v\rangle m)$ and consider $g_t = e^{\mathcal{B}_{\lambda,\delta} t}(\partial^\beta_x h_0)$, for any $|\beta| \leq s$, which satisfies (using the fact that the x-derivatives commute with the semigroup)

$$\partial_t g_t + v\cdot\nabla_x g_t = Q_{e_\lambda}(G_\lambda, g_t) + Q_{e_\lambda}(g_t, G_\lambda) + \lambda^\gamma \Delta_v g_t - \mathcal{A}_\delta g_t.$$

Let us define $D_t := t\nabla_x + \nabla_v$. D_t commute with the free transport equation and the Laplacian Δ_v. Using these properties of commutativity and the property (4.17) of the collision operator, we have

$$\begin{aligned}\partial_t(D_t g_t) + v\cdot\nabla_x(D_t g_t) = {}& Q_{e_\lambda}(\nabla_v G_\lambda, g_t) + Q_{e_\lambda}(g_t, \nabla_v G_\lambda) + Q_{e_\lambda}(G_\lambda, D_t g_t)\\ &+ Q_{e_\lambda}(D_t g_t, G_\lambda) + \lambda^\gamma\Delta_v g_t - D_t(\mathcal{A}_\delta g_t).\end{aligned}$$

With the notations of (4.15), we rewrite the last term as

$$\begin{aligned}D_t(\mathcal{A}_\delta g_t)(v) &= D_t\int_{\mathbb{R}^3} k_\delta(v,v_*)\, g_t(v_*)\, dv_*\\ &= \int_{\mathbb{R}^3}\nabla_v k_\delta(v,v_*)\, g_t(v_*)\, dv_* - \int_{\mathbb{R}^3} k_\delta(v,v_*)\,\nabla_{v_*} g_t(v_*)\, dv_*\\ &\quad + \int_{\mathbb{R}^3} k_\delta(v,v_*)\,(D_t g_t)(v_*)\, dv_*\\ &= \mathcal{A}^1_\delta g_t + \mathcal{A}^2_\delta g_t + \mathcal{A}_\delta(D_t g_t),\end{aligned}$$

where $\mathcal{A}^1_\delta$ stands for the integral operator associated to the kernel $\nabla_v k_\delta$ and $\mathcal{A}^2_\delta$ stands for the integral operator associated to the kernel $\nabla_{v_*} k_\delta$. All together, we may write

$$\partial_t (D_t g_t) = \mathcal{B}_{\lambda,\delta}(D_t g_t) + \mathcal{I}_\delta(g_t)$$

with

$$\mathcal{I}_\delta f = Q_{e_\lambda}(\nabla_v G_\lambda, f) + Q_{e_\lambda}(f, \nabla_v G_\lambda) - \mathcal{A}^1_\delta f - \mathcal{A}^2_\delta f,$$

which satisfies

$$\|\mathcal{I}_\delta f\|_{L^1_v(\langle v\rangle m)} \le C_\delta \|f\|_{L^1_v(\langle v\rangle^2 m)}.$$

Then arguing as in the proof of Lemma 4.2.76, we obtain, for any $\alpha_0'' \in (0, \alpha_0)$ and for η small enough

$$\frac{d}{dt}\left(e^{\alpha_0'' t}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} (\eta |D_t g_t| + |g_t|)\langle v\rangle m\, dx\, dv\right) \le 0,$$

which implies

$$\forall\, t \ge 0, \quad \|D_t g_t\|_{L^1(\langle v\rangle m)} + \|g_t\|_{L^1(\langle v\rangle m)} \le \eta^{-1} e^{-\alpha_0'' t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \tag{4.29}$$

Then, we write

$$\begin{aligned} t\,\nabla_x T_1(t)(\partial_x^\beta h_0) &= \int_{\mathbb{R}^3} k_\delta(v, v_*)\,[(D_t g_t) - \nabla_{v_*} g_t]\,(x, v_*)\, dv_* \\ &= \mathcal{A}_\delta(D_t g_t) + \mathcal{A}^2_\delta g_t, \end{aligned}$$

Using (4.29), we hence get

$$\begin{aligned} t\,\|\nabla_x T_1(t)(\partial_x^\beta h_0)\|_{L^1(K)} &\le C\,\left(\|D_t g_t\|_{L^1(\langle v\rangle m)} + \|g_t\|_{L^1(\langle v\rangle m)}\right) \\ &\le C\,\eta^{-1} e^{-\alpha_0'' t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \end{aligned}$$

Together with estimate (4.28) and Lemma 4.2.77, for $s \ge 0$, we conclude that

$$\|T_1(t)(\partial_x^\beta h_0)\|_{W^{1,1}_x W^{s+1,1}_v(K)} \le \frac{C e^{-\alpha_0'' t}}{t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)},$$

which in turn implies (4.26).

Now interpolating the last inequality and (4.28), for $s \ge 0$, we have

$$\|T_1(t) h_0\|_{W^{s+1/2,1}_{x,v}(K)} \le \frac{C e^{-\alpha_0'' t}}{\sqrt{t}}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \tag{4.30}$$

Putting together (4.26) and (4.30), for $s \ge 0$, we obtain

$$\begin{aligned} \|T_2(t) h_0\|_{W^{s+1/2,1}_{x,v}(K)} &\le \int_0^t \|T_1(t-s)T_1(s)h_0\|_{W^{s+1/2,1}_{x,v}(K)}\, ds \\ &\le C\int_0^t \frac{e^{-\alpha_0''(t-s)}}{(t-s)^{1/2}}\|T_1(s)h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}\, ds \\ &\le C\left(\int_0^t \frac{e^{-\alpha_0''(t-s)}}{(t-s)^{1/2}} e^{-\alpha_0 s}\, ds\right)\|h_0\|_{W^{s,1}_{x,v}(\langle v\rangle m)} \\ &\le C\sqrt{t} e^{-\alpha_0'' t}\|h_0\|_{W^{s,1}_{x,v}(\langle v\rangle m)}, \end{aligned}$$

which concludes the proof. □

Let us now recall [51, Lemma 2.17] which yields an estimate on the norms $\|T_n\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})}$ for $j = -1, 0$.

Lemma 4.2.79. *Let E, $\mathcal{E}$ be two Banach space with $E \subset \mathcal{E}$ dense with continuous embedding, and consider $\mathcal{L} \in \mathcal{E}$ and $a \in \mathbb{R}$. We assume that there exist some intermediate spaces*

$$E = \mathcal{E}_J \subset \mathcal{E}_{J-1} \subset ... \subset \mathcal{E}_2 \subset \mathcal{E}_1 = \mathcal{E}, \quad J \geq 2$$

such that, denoting $\mathcal{A}_j := \mathcal{A}_{|\mathcal{E}_j}$ and $\mathcal{B}_j := \mathcal{B}_{|\mathcal{E}_j}$

(i) *$(\mathcal{B}_j - a)$ is hypodissipative and $\mathcal{A}_j$ is bounded on $\mathcal{E}_j$ for $1 \leq j \leq J$;*

(ii) *there are some constants $\ell_0 \in \mathbb{N}^*$, $C \geq 1$, $K \in \mathbb{R}$, $\gamma \in [0,1)$ such that*

$$\forall\, t \geq 0, \quad \|T_{\ell_0}(t)\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})} \leq C \frac{e^{Kt}}{t^\gamma},$$

for $1 \leq j \leq J-1$, with the notation $T_\ell := (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}$.

Then for any $a' > a$, there exist some constructive constants $n \in \mathbb{N}$, $C_{a'} \geq 1$ such that

$$\forall\, t \geq 0, \quad \|T_n(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq C_{a'} e^{a't}.$$

Combining Lemmas 4.2.76 and 4.2.78, we can apply Lemma 4.2.79 and deduce the following result:

Lemma 4.2.80. *Let λ be in $(0, \lambda_0)$. For any $\alpha_0' \in (0, \alpha_0)$, there exist some constructive constants $n \in \mathbb{N}$ and $C_{\alpha_0'} \geq 1$ such that*

$$\forall\, t \geq 0, \quad \|T_n(t)\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})} \leq C_{\alpha_0'} e^{-\alpha_0' t}, \quad j = -1, 0.$$

4.2.6 Estimate on $\mathcal{L}_\lambda - \mathcal{L}_0$

Using estimates from the proof of Lemma 4.2.76, we can prove the following result:

Lemma 4.2.81. *There exists a function $\eta_1(\lambda)$ such that $\eta_1(\lambda) \xrightarrow[\lambda \to 0]{} 0$ and the difference $\mathcal{L}_\lambda - \mathcal{L}_0$ satisfies*

$$\|\mathcal{L}_\lambda - \mathcal{L}_0\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j-1})} \leq \eta_1(\lambda), \;\; j = 0, 1.$$

Proof. We have

$$\mathcal{L}_\lambda - \mathcal{L}_0 = \lambda^\gamma \Delta_v + \widehat{Q}_{e_\lambda} - \widehat{Q}_1.$$

First, we have the following inequality:

$$\|\, \lambda^\gamma \Delta_v(h)\|_{\mathcal{E}_{j-1}} \leq \lambda^\gamma \|h\|_{\mathcal{E}_j}, \quad j = 0, 1. \tag{4.31}$$

Concerning the term $\widehat{Q}_{e_\lambda} - \widehat{Q}_1$, we have obtained in the proof of Lemma 4.2.76

$$\|(\widehat{Q}_{e_\lambda} - \widehat{Q}_1)h\|_{L^1_v(\langle v\rangle m)} \leq C\, \varepsilon(\lambda) \|h\|_{L^1_v(\langle v\rangle^2 m)}$$

with $\varepsilon(\lambda) \xrightarrow[\lambda \to 0]{} 0$. Again arguing as in the proof of Lemma 4.2.76, we obtain

$$\|\partial_v(\widehat{Q}_{e_\lambda} - \widehat{Q}_1)h\|_{L^1_v(\langle v \rangle m)} \leq C\,\varepsilon(\lambda)\|h\|_{W^{1,1}_v(\langle v \rangle^2 m)}.$$

We obtain the higher-order derivatives in the same way and we can conclude that

$$\|(\widehat{Q}_{e_\lambda} - \widehat{Q}_1)h\|_{\mathcal{E}_0} \leq C\,\varepsilon(\lambda)\|h\|_{\mathcal{E}_1}. \tag{4.32}$$

Gathering (4.31) and (4.32), we deduce that

$$\|(\mathcal{L}_\lambda - \mathcal{L}_0)h\|_{\mathcal{E}_0} \leq \eta_1(\lambda)\|h\|_{\mathcal{E}_1}.$$

Using the same method, we obtain:

$$\|(\mathcal{L}_\lambda - \mathcal{L}_0)h\|_{\mathcal{E}_{-1}} \leq \eta_1(\lambda)\|h\|_{\mathcal{E}_0}.$$

□

In the remaining part of the paper, δ is fixed (given by Lemma 4.2.76), we hence denote $\mathcal{A} = \mathcal{A}_\delta$ and $\mathcal{B}_\lambda = \mathcal{B}_{\lambda,\delta}$.

4.2.7 Semigroup spectral analysis of the linearized operator

In this section we shall state some results on the geometry of the spectrum of the linearized diffusive inelastic collision operator for a small diffusion parameter.

Theorem 4.2.82. *There exists $\lambda' \in [0,1)$ such that for any $\lambda \in [0,\lambda']$, $\mathcal{L}_\lambda$ satisfies the following properties in $\mathcal{E}_0$:*

(i) *There exists $\mu_\lambda \in \mathbb{R}$ such that $\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} = \{\mu_\lambda, 0\}$ where α is given by Theorem 4.2.83. Moreover, 0 is a four-dimensional eigenvalue and μ_λ is a one-dimensional eigenvalue.*

(ii) *μ_λ satisfies the following estimate*

$$\mu_\lambda = -C\lambda^\gamma + o(\lambda^\gamma) \tag{4.33}$$

for some $C > 0$.

(iii) *For any $\alpha' \in (0, \min(\alpha,\alpha_0)) \setminus \{-\mu_\lambda\}$ (where α_0 is provided by Lemma 4.2.76), the semigroup generated by $\mathcal{L}_\lambda$ has the following decay property*

$$\forall\, t \geq 0, \quad \|e^{\mathcal{L}_\lambda t} - e^{\mathcal{L}_\lambda t}\Pi_{\mathcal{L}_\lambda,0} - e^{\mathcal{L}_\lambda t}\Pi_{\mathcal{L}_\lambda,\mu_\lambda}\|_{\mathscr{B}(\mathcal{E}_0)} \leq Ce^{-\alpha' t} \tag{4.34}$$

for some $C > 0$.

The proof is divided into several steps.

Step 1 of the proof: the linearized elastic operator

We recall hypodissipativity results for the semigroup associated to the linearized elastic Boltzmann equation which are proved in [51]. Among other things, the following is proved in this paper (Theorem 4.2):

Theorem 4.2.83. *There are constructive constants $C \geq 1$, $\alpha > 0$, such that the operator $\mathcal{L}_0$ satisfies in $\mathcal{E}_0$ and $\mathcal{E}_1$:*

$$\Sigma(\mathcal{L}_0) \cap \Delta_{-\alpha} = \{0\} \quad \text{and} \quad N(\mathcal{L}_0) = Span\{G_0, v_1 G_0, v_2 G_0, v_3 G_0, |v|^2 G_0\}.$$

Moreover, $\mathcal{L}_0$ is the generator of a strongly continuous semigroup $h(t) = S_{\mathcal{L}_0}(t) h_{in}$ in $\mathcal{E}_0$ and $\mathcal{E}_1$, solution to the initial value problem (4.14) with $\lambda = 0$, which satisfies:

$$\forall\, t \geq 0, \quad \|h(t) - \Pi_{\mathcal{L}_0,0} h_{in}\|_{\mathcal{E}_i} \leq C e^{-\alpha t} \|h_{in} - \Pi_{\mathcal{L}_0,0} h_{in}\|_{\mathcal{E}_i}, \quad i = 0, 1.$$

Step 2 of the proof: localization of spectrum of $\mathcal{L}_\lambda$

Lemma 4.2.84. *Let us define $K_\lambda(z)$ for any $z \in \Omega := \Delta_{-\alpha} \setminus \{0\}$ (where α is given by Theorem 4.2.83) by*

$$K_\lambda(z) = (-1)^n \, (\mathcal{L}_\lambda - \mathcal{L}_0) \, \mathcal{R}_{\mathcal{L}_0}(z) \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^n.$$

Then, there exists $\eta_2(\lambda)$ with $\eta_2(\lambda) \xrightarrow[\lambda \to 0]{} 0$ such that

$$\forall\, z \in \Omega_\lambda := \Delta_{-\alpha} \setminus \bar{B}(0, \eta_2(\lambda)), \quad \|K_\lambda(z)\|_{\mathscr{B}(\mathcal{E}_0)} \leq \eta_2(\lambda).$$

Moreover, there exists $\lambda' \in (0, \lambda_0]$ (where λ_0 is given by Lemma 4.2.76) such that for any $\lambda \in [0, \lambda']$, we have

(i) *$I + K_\lambda(z)$ is invertible for any $z \in \Omega_\lambda$*

(ii) *$\mathcal{L}_\lambda - z$ is also invertible for any $z \in \Omega_\lambda$ and*

$$\forall\, z \in \Omega_\lambda, \quad \mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{U}_\lambda(z) \, (I + K_\lambda(z))^{-1}$$

where

$$\mathcal{U}_\lambda(z) = \mathcal{R}_{\mathcal{B}_\lambda}(z) + \ldots + (-1)^{n-1} \, \mathcal{R}_{\mathcal{B}_\lambda}(z) \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^{n-1} + (-1)^n \, \mathcal{R}_{\mathcal{L}_0} \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^n.$$

We thus deduce that

$$\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} \subset B(0, \eta_2(\lambda)).$$

Proof. Step 1. We first notice that $(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^n \in \mathscr{B}(\mathcal{E}_0, \mathcal{E}_1)$, $\mathcal{R}_{\mathcal{L}_0}(z) \in \mathscr{B}(\mathcal{E}_1)$ and $\mathcal{L}_\lambda - \mathcal{L}_0 \in \mathscr{B}(\mathcal{E}_1, \mathcal{E}_0)$ for any $z \in \Omega$ because of Lemma 4.2.80, Theorem 4.2.83 and Lemma 4.2.81. Moreover, there exist $n \in \mathbb{N}$ and a constant $C_0 > 0$ such that

$\|\mathcal{R}_{\mathcal{L}_0}(z)\|_{\mathscr{B}(\mathcal{E}_1)} \le C_0/|z|^n$ for any z in Ω. Indeed, we know from [57, paragraph I.5.3] that in $\mathcal{E}_1$, the following Laurent series

$$\mathcal{R}_{\mathcal{L}_0}(z) = \sum_{k=-n}^{+\infty} z^k \, \mathcal{C}_k$$

where $\mathcal{C}_k$ are some bounded operators in $\mathscr{B}(\mathcal{E}_1)$, converges for z close to 0. We thus deduce the previous estimate on $\|\mathcal{R}_{\mathcal{L}_0}(z)\|_{\mathscr{B}(\mathcal{E}_1)}$. Let us finally define $\eta_2(\lambda) := \left(C_0 C_{\lambda_0'} \eta_1(\lambda)\right)^{\frac{1}{n+1}}$ where λ_0' is fixed in $(0, \lambda_0)$ and $C_{\lambda_0'}$ is given by Lemma 4.2.80. We deduce that

$$\forall\, z \in \Omega_\lambda, \quad \|K_\lambda(z)\|_{\mathscr{B}(\mathcal{E}_0)} \le \eta_1(\lambda) \frac{C_0}{\eta_2(\lambda)^n} C_{\lambda_0'} = \eta_2(\lambda).$$

We then choose $\lambda' \in (0, \lambda_0]$ such that for any $\lambda \in (0, \lambda']$, $\eta_2(\lambda) < 1$. We hence obtain that $I + K_\lambda(z)$ is an invertible operator for any $\lambda \in (0, \lambda']$. Let us now consider $\lambda \in (0, \lambda']$.

Step 2. $\mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ is a right-inverse of $\mathcal{L}_\lambda - z$ on Ω_λ. For any $z \in \Omega_\lambda$, we compute

$$\begin{aligned}
(\mathcal{L}_\lambda - z)\,\mathcal{U}_\lambda(z) \;&=\; (\mathcal{B}_\lambda - z + \mathcal{A})\,\{\mathcal{R}_{\mathcal{B}_\lambda}(z) + \ldots + (-1)^{n-1}\,\mathcal{R}_{\mathcal{B}_\lambda}(z)\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda})^{n-1}(z)\} \\
&+\; (-1)^n\,(\mathcal{L}_\lambda - \mathcal{L}_0 + \mathcal{L}_0 - z)\,\mathcal{R}_{\mathcal{L}_0}(z)\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda})^n(z) \\
&=\; Id + K_\lambda(z).
\end{aligned}$$

Because of the previous step, we deduce that for $z \in \Omega_\lambda$, $\mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ is a right-inverse of $\mathcal{L}_\lambda - z$.

Step 3. There exists $z_0 \in \Omega_\lambda$ such that $\mathcal{L}_\lambda - z_0$ is invertible on Ω_λ. Indeed, we write

$$\mathcal{L}_\lambda - z_0 = (\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda}(z_0) + I)\,(\mathcal{B}_\lambda - z_0)$$

where $(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda}(z_0) + I)$ is invertible for $\Re e\, z_0$ large enough because of Lemma 4.2.76. As a consequence, $\mathcal{L}_\lambda - z_0$ is the product of two invertible operators, we hence obtain that $\mathcal{L}_\lambda - z_0$ is invertible.

Step 4. $\mathcal{L}_\lambda - z$ is invertible close to z_0. Since $\mathcal{L}_\lambda - z_0$ is invertible on Ω_λ, we have $\mathcal{R}_{\mathcal{L}_\lambda}(z_0) = \mathcal{U}_\lambda(z_0)\,(I + K_\lambda(z_0))^{-1}$. Moreover, if $\|\mathcal{R}_{\mathcal{L}_\lambda}(z_0)\| \le C$ for some $C > 0$, then $\mathcal{L}_\lambda - z$ in invertible on the disc $B(z_0, 1/C)$ with

$$\forall\, z \in B(z_0, 1/C), \quad \mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{R}_{\mathcal{L}_\lambda}(z_0) \sum_{n=0}^{+\infty} (z - z_0)^n\, \mathcal{R}_{\mathcal{L}_\lambda}(z_0)^n, \tag{4.35}$$

and arguing as before, $\mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ on $B(z_0, 1/C)$ since $\mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ is a right inverse of $\mathcal{L}_\lambda - z$ for any $z \in \Omega_\lambda$.

Step 5. $\mathcal{L}_\lambda - z$ is invertible on Ω_λ. For a given $z_1 \in \Omega_\lambda$, we consider a continuous path Γ from z_0 to z_1 included in Ω_λ, i.e. a continuous function $\Gamma : [0,1] \to \Omega_\lambda$ such that

$\Gamma(0) = z_0$, $\Gamma(1) = z_1$. We know that $(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^\ell$, $1 \le \ell \le n-1$, $\mathcal{R}_{\mathcal{L}_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^n$ and $(I + K_\lambda(z))^{-1}$ are locally uniformly bounded in $\mathscr{B}(\mathcal{E}_0)$ on Ω_λ, which implies

$$\sup_{z\in\Gamma([0,1])} \|\mathcal{U}_\lambda(z)(I+K_\lambda(z))^{-1}\|_{\mathscr{B}(\mathcal{E}_0)} := K < \infty.$$

Since $(\mathcal{L}_\lambda - z_0)$ is invertible we deduce that $(\mathcal{L}_\lambda - z)$ is invertible with $\mathcal{R}_{\mathcal{L}_\lambda}(z)$ locally bounded around z_0 with a bound K which is uniform along Γ (and a similar series expansion as in (4.35)). By a continuation argument we hence obtain that $(\mathcal{L}_\lambda - z)$ is invertible in $\mathcal{E}_0$ all along the path Γ with

$$\mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{U}_\lambda(z)(I+K_\lambda(z))^{-1} \quad \text{and} \quad \|\mathcal{R}_{\mathcal{L}_\lambda}(z)\|_{\mathscr{B}(\mathcal{E}_0)} \le K.$$

Hence we conclude that $(\mathcal{L}_\lambda - z_1)$ is invertible with $\mathcal{R}_{\mathcal{L}_\lambda}(z_1) = \mathcal{U}_\lambda(z_1)(I+K_\lambda(z_1))^{-1}$. □

Step 3 of the proof: dimension of eigenspaces

Lemma 4.2.85. *There exist a constant $C > 0$ and a function $\eta_3(\lambda)$ such that*

$$\|\Pi_{\mathcal{L}_\lambda,-\alpha}\|_{\mathscr{B}(\mathcal{E}_0,\mathcal{E}_1)} \le C, \tag{4.36}$$

and

$$\|\Pi_{\mathcal{L}_\lambda,-\alpha} - \Pi_{\mathcal{L}_0,-\alpha}\|_{\mathscr{B}(\mathcal{E}_0)} \le \eta_3(\lambda), \quad \eta_3(\lambda) \xrightarrow[\lambda\to 0]{} 0. \tag{4.37}$$

It implies that for λ close enough to 0, we have

$$\dim \mathrm{R}(\Pi_{\mathcal{L}_\lambda,-\alpha}) = \dim \mathrm{R}(\Pi_{\mathcal{L}_0,-\alpha}) = 5.$$

The following lemma from [57, paragraph I.4.6] is going to be useful for the proof.

Lemma 4.2.86. *Let X be a Banach space and P, Q be two projectors in $\mathscr{B}(X)$ such that $\|P - Q\|_{\mathscr{B}(X)} < 1$. Then the ranges of P and Q are isomorphic. In particular, $\dim(\mathrm{R}(P)) = \dim(\mathrm{R}(Q))$.*

Let us now prove Lemma 4.2.85.

Proof. Let $\Gamma := \{z \in \mathbb{C}, |z| = \eta_2(\lambda)\}$ which is included in Ω for λ small enough. We set $N := 2n$ and we define

$$\mathcal{U}_\lambda^0 := \mathcal{R}_{\mathcal{B}_\lambda} + ... + (-1)^{N-1}\,\mathcal{R}_{\mathcal{B}_\lambda}\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda})^{N-1} \text{ and } \quad \mathcal{U}_\lambda^1 := (-1)^N\,\mathcal{R}_{\mathcal{L}_0}\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda})^N,$$

Notice that Lemma 4.2.76 implies that $z \mapsto \mathcal{R}_{\mathcal{B}_\lambda}(z)$ is holomorphic in $\bar{B}(0, \eta_2(\lambda))$ for λ small enough and consequently that $\int_\Gamma \mathcal{U}_\lambda^0(z)\, dz = 0$. We can then compute:

$$\begin{aligned}
\Pi_{\mathcal{L}_\lambda,-\alpha} &= \frac{i}{2\pi} \int_\Gamma \mathcal{R}_{\mathcal{L}_\lambda}(z)\, dz \\
&= \frac{i}{2\pi} \int_\Gamma \mathcal{U}_\lambda(z)\, (I + K_\lambda(z))^{-1}\, dz \\
&= \frac{i}{2\pi} \int_\Gamma \mathcal{U}_\lambda^0(z)\, \{I - K_\lambda(z)\, (I + K_\lambda(z))^{-1}\}\, dz \\
&\quad + \frac{i}{2\pi} \int_\Gamma \mathcal{U}_\lambda^1(z)\, (I + K_\lambda(z))^{-1}\, dz \\
&= \frac{1}{2i\pi} \int_\Gamma \mathcal{U}_\lambda^0(z)\, K_\lambda(z)\, (I + K_\lambda(z))^{-1}\, dz \\
&\quad + (-1)^n\, \frac{i}{2\pi} \int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^N\, (I + K_\lambda(z))^{-1}\, dz.
\end{aligned}$$

Since $(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^N$ appears in the two parts of the expression of $\Pi_{\mathcal{L}_\lambda,-\alpha}$, we deduce that (4.36) holds.

Concerning the estimate on $\Pi_{\mathcal{L}_0,-\alpha} - \Pi_{\mathcal{L}_\lambda,-\alpha}$, we begin by writing

$$\mathcal{R}_{\mathcal{L}_0}(z) = \mathcal{R}_{\mathcal{B}_0}(z) + ... + (-1)^{N-1}\, \mathcal{R}_{\mathcal{B}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^{N-1} + (-1)^N\, \mathcal{R}_{\mathcal{L}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^N$$

which implies that

$$\begin{aligned}
\Pi_{\mathcal{L}_0,-\alpha} &= \frac{i}{2\pi} \int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, dz \\
&= (-1)^n\, \frac{i}{2\pi} \int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^N\, dz.
\end{aligned}$$

Finally, we deduce that

$$\begin{aligned}
&\Pi_{\mathcal{L}_0,-\alpha} - \Pi_{\mathcal{L}_\lambda,-\alpha} \\
= \ &(-1)^n\, \frac{i}{2\pi} \int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, \{(\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^N - (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^N\, (I + K_\lambda(z))^{-1}\}\, dz \\
&- \frac{1}{2i\pi} \int_\Gamma \mathcal{U}_\lambda^0(z)\, K_\lambda(z)\, (I + K_\lambda(z))^{-1}\, dz.
\end{aligned}$$

Since $K_\lambda(z)$ appears in the second term, we deduce that it is bounded by $\eta_2(\lambda)$. Concerning the first term, we rewrite it as

$$(\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^{2n} - (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n} + (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n}(I - (I + K_\lambda(z))^{-1}).$$

The second part of this expression is bounded by $\eta_2(\lambda)/(1 - \eta_2(\lambda))$ because of the bound on the norm of K_λ. The first part can be written as

$$\sum_{k=0}^{2n} (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^k\, \mathcal{A}\, (\mathcal{R}_{\mathcal{B}_0}(z) - \mathcal{R}_{\mathcal{B}_\lambda}(z))\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n-k-1}.$$

In addition, the bound on the norm of $\mathcal{B}_\lambda - \mathcal{B}_0$ given by Lemma 4.2.81 gives a bound on the norm of $\mathcal{R}_{\mathcal{B}_\lambda}(z) - \mathcal{R}_{\mathcal{B}_0}(z)$ because

$$\mathcal{R}_{\mathcal{B}_1}(z) - \mathcal{R}_{\mathcal{B}_\lambda}(z) = \mathcal{R}_{\mathcal{B}_\lambda}(z)\,(\mathcal{B}_\lambda - \mathcal{B}_0)\,\mathcal{R}_{\mathcal{B}_0}(z).$$

Since for all k, $0 \le k \le 2n$ we have $k \ge n$ or $2n-k-1 \ge n$, we can use Lemma 4.2.80 and conclude that $(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_0}(z))^{2n} - (\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n}$ is bounded by $C\eta_1(\lambda)$, which concludes the proof of (4.37).

The last part of Lemma 4.2.85 is nothing but Lemma 4.2.86 because for λ close enough to 0, $\eta_3(\lambda) < 1$. □

We can now finish the proof of Theorem 4.2.82-**(i)**. The previous lemma implies that there exist $\xi_1, ..., \xi_5 \in \mathbb{C}$ such that

$$\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} = \{\xi_1, ...\xi_5\}.$$

Moreover, we know that 0 is a four-dimensional eigenvalue due to the conservation of mass and momentum. Since the operator is real, we can deduce that there exists $\mu_\lambda \in \mathbb{R}$ such that

$$\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} = \{0, \mu_\lambda\}.$$

Step 4 of the proof: fine study of spectrum close to 0

Concerning the case of a constant coefficient of inelasticity, we refer to [67, Section 5.2, Step 2] for the proof of Theorem 4.2.82-**(ii)** (the first order expansion of μ_λ (4.33)). Let us deal with the non-constant case.

We first denote ϕ_0 the energy eigenfunction of the the elastic linearized operator associated to 0 such that $\|\phi_0\|_{L^1_v(\langle v\rangle^2)} = 1$. We also denote Π_0 the projection on $\mathbb{R}\,\phi_0$ and $\pi_0\psi$ the coordinate of $\Pi_0\,\psi$ on $\mathbb{R}\,\phi_0$ i.e $\Pi_0\,\psi = (\pi_0\psi)\,\phi_0$. Finally, we denote ϕ_λ the unique eigenfunction associated to μ_λ such that $\|\phi_\lambda\|_{L^1_v(\langle v\rangle^2)} = 1$ and $\pi_0\phi_\lambda \ge 0$.

By integrating in v the eigenvalue equation related to μ_λ

$$\mathcal{L}_\lambda\,\phi_\lambda = \mu_\lambda\,\phi_\lambda$$

against $|v|^2$, we get

$$2\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(G_\lambda, \phi_\lambda)\,|v|^2\,dv + \lambda^\gamma \int_{\mathbb{R}^3} \Delta_v\,\phi_\lambda\,|v|^2\,dv = \mu_\lambda\,\mathcal{E}(\phi_\lambda). \tag{4.38}$$

We now compute the left-hand side of (4.38). By a classical computation which uses (4.9), we have:

$$2\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(G_\lambda, \phi_\lambda)\,|v|^2\,dv = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |u|^3 G_{\lambda *}\phi_\lambda\,\frac{1-\widehat{u}\cdot\sigma}{4}(1-{e_\lambda}^2)\,d\sigma\,dv_*\,dv$$

and using polar coordinates

$$\int_{\mathbb{S}^2} \frac{1-\widehat{u}\cdot\sigma}{4}\left(1-{e_\lambda}^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)d\sigma = 4\pi\int_0^1 \Big(1-{e_\lambda}^2\,(|u|y)\Big)\,y^3\,dy.$$

Let us define
$$\psi_e(r) := 4\pi\, r^{3/2} \int_0^1 (1 - e^2(\sqrt{r}z))\, z^3\, dz,$$
we can compute $\psi_{e_\lambda}(r) = \lambda^{-3}\, \psi_e(\lambda^2 r)$. We deduce that
$$2\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(G_\lambda, \phi_\lambda)\, |v|^2\, dv = -\frac{1}{\lambda^3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*} \phi_\lambda\, \psi_e(\lambda^2 |u|^2)\, dv_*\, dv.$$
We also have
$$\int_{\mathbb{R}^3} \Delta_v\, \phi_\lambda\, |v|^2\, dv = 6\int_{\mathbb{R}^3} \phi_\lambda\, dv = 6\,\rho(\phi_\lambda).$$

Dividing (4.38) by λ^γ, we hence obtain
$$-\frac{1}{\lambda^{3+\gamma}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\phi_\lambda\, \psi_e(\lambda^2|u|^2)\, dv_*\, dv + 6\,\rho(\phi_\lambda) = \frac{1}{\lambda^\gamma}\, \mu_\lambda\, \mathcal{E}(\phi_\lambda). \tag{4.39}$$

We would like to make λ tend to 0 in (4.39). To do that, we introduce the following notations:
$$I_\lambda(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_* g\, \zeta_\lambda(|u|^2)\, dv_*\, dv \quad \text{with} \quad \zeta_\lambda(r^2) = \frac{1}{\lambda^{3+\gamma}}\, \psi_e(\lambda^2 r^2),$$
and
$$I_0(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_* g\, \zeta_0(|u|^2)\, dv_*\, dv \quad \text{with} \quad \zeta_0(r^2) = \frac{a}{4+\gamma}\, r^{3+\gamma}.$$
Let us now prove that $I_\lambda(G_\lambda, \phi_\lambda)$ tends to $I_0(G_0, \phi_0)$ as λ tends to 0. We state the following lemma which is going to be useful. We do not prove it here because the proof is the same as the one of [67, Lemma 5.17].

Lemma 4.2.87. *Let k, $q \in \mathbb{N}$. We have the following result:*
$$\|\, \phi_\lambda - \phi_0\|_{W^{k,1}_v(\langle v\rangle^q m)} \xrightarrow[\lambda\to 0]{} 0.$$

To prove that $I_\lambda(G_\lambda, \phi_\lambda)$ tends to $I_0(G_0, \phi_0)$ as λ tends to 0, let us write the following inequality:
$$\begin{aligned} |I_\lambda(G_\lambda,\phi_\lambda) - I_0(G_0,\phi_0)| &\le |I_\lambda(G_\lambda,\phi_\lambda) - I_0(G_\lambda,\phi_\lambda)| + |I_0(G_\lambda,\phi_\lambda) - I_0(G_0,\phi_0)| \\ &=: J^1_\lambda + J^2_\lambda. \end{aligned}$$

We first deal with J^2_λ:
$$\begin{aligned} J^2_\lambda &= \left| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (G_{\lambda_*}\phi_\lambda - G_{0_*}\phi_0)\, \zeta_0(|u|^2)\, dv_*\, dv \right| \\ &\le C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |G_{\lambda_*} - G_{0_*}|\, \phi_0\, \langle v\rangle^{3+\gamma}\, \langle v_*\rangle^{3+\gamma}\, dv_*\, dv \\ &\quad + C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\, |\phi_\lambda - \phi_0|\;\, \langle v\rangle^{3+\gamma}\, \langle v_*\rangle^{3+\gamma}\, dv_*\, dv \\ &\le C\left(\|G_\lambda - G_0\|_{L^1_v(\langle v\rangle^{3+\gamma})}\, \|\, \phi_0\|_{L^1_v(\langle v\rangle^{3+\gamma})} + \|G_\lambda\|_{L^1_v(\langle v\rangle^{3+\gamma})}\, \|\phi_\lambda - \phi_0\|_{L^1_v(\langle v\rangle^{3+\gamma})} \right) \\ &\le C\left(\|G_\lambda - G_0\|_{L^1_v(\langle v\rangle^{3+\gamma})} + \|\phi_\lambda - \phi_0\|_{L^1_v(\langle v\rangle^{3+\gamma})} \right) \xrightarrow[\lambda\to 0]{} 0 \end{aligned}$$

because of Lemmas 4.2.72 and 4.2.87.

Let us now establish an estimate on J^1_λ:

$$J^1_\lambda \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda *}\phi_\lambda \, |\zeta_\lambda(|u|^2) - \zeta_0(|u|^2)| \, dv_* \, dv =: D_\lambda.$$

We can rewrite the difference $\zeta_\lambda(r^2) - \zeta_0(r^2)$ in the following way:

$$\zeta_\lambda(r^2) - \zeta_0(r^2) = \frac{r^{3+\gamma}}{2} \int_0^1 \left(\frac{1 - e^2(\lambda r z)}{(\lambda r z)^\gamma} - 2a \right) z^{3+\gamma} \; dz,$$

which allows us to get an estimate on this difference using Assumption 4.1.67-(3). There exists a constant $C > 0$ such that

$$\forall \, \lambda \in (0,1], \quad \forall r > 0, \quad |\zeta_\lambda(r^2) - \zeta_0(r^2)| \leq C \left(r^{3+2\gamma} \, \lambda^\gamma + r^{3+\gamma+\overline{\gamma}} \, \lambda^{\overline{\gamma}} + r^{3+\overline{\gamma}} \, \lambda^{\overline{\gamma}-\gamma} \right).$$

Denoting $\widetilde{\gamma} := \min(\gamma, \overline{\gamma} - \gamma)$, we can deduce that

$$\begin{aligned} D_\lambda &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda *}\phi_\lambda \, \lambda^{\widetilde{\gamma}} |u|^{3+\gamma+\overline{\gamma}} \; dv_* \, dv \\ &\leq C \, \lambda^{\widetilde{\gamma}} \, \|G_\lambda\|_{L^1_v(\langle v \rangle^{3+\gamma+\overline{\gamma}})} \|\phi_\lambda\|_{L^1_v(\langle v \rangle^{3+\gamma+\overline{\gamma}})} \\ &\leq C \, \lambda^{\widetilde{\gamma}}. \end{aligned}$$

It yields the result: $J^1_\lambda \xrightarrow[\lambda \to 0]{} 0$.

We can now make λ tend to 0 in (4.39). Using the previous result $I_\lambda(G_\lambda, \phi_\lambda) \to I_0(G_0, \phi_0)$, the fact that the mass of ϕ_0 is 0 and the convergences $G_\lambda \to G_0$ and $\phi_\lambda \to \phi_0$ (Lemmas 4.2.72 and 4.2.87), we deduce that

$$\frac{\mu_\lambda}{\lambda^\gamma} \mathcal{E}(\phi_0) = -I_0(G_0, \phi_0) + o(1).$$

We finally conclude that there exists a constant $C > 0$ such that

$$\mu_\lambda = -C \lambda^\gamma + o(\lambda^\gamma).$$

Step 5 of the proof: semigroup decay

In order to get our semigroup decay, we are going to apply the following quantitative spectral mapping theorem which comes from [77]. We give here a simpler version and hence give the proof which is easier in this case.

Proposition 4.2.88. *Consider a Banach space X and an operator $\Lambda \in \mathscr{C}(X)$ so that $\Lambda = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \in \mathscr{B}(X)$ and $\mathcal{B} - a$ is hypodissipative on X for some $a \in \mathbb{R}$. We assume furthermore that there exists a family X_j, $1 \leq j \leq m$, $m \geq 2$ of intermediate spaces such that*

$$X_m \subset \mathrm{D}(\Lambda^2) \subset X_{m-1} \subset ... \subset X_2 \subset X_1 = X,$$

and a family of operators $\Lambda_j, \mathcal{A}_j, \mathcal{B}_j \in \mathscr{C}(X_j)$ *such that*

$$\Lambda_j = \mathcal{A}_j + \mathcal{B}_j, \quad \Lambda_j = \Lambda_{|X_j}, \quad \mathcal{A}_j = \mathcal{A}_{|X_j}, \quad \mathcal{B}_j = \mathcal{B}_{|X_j},$$

and that there holds

(i) $(\mathcal{B}_j - a)$ *is hypodissipative on* X_j*;*

(ii) $\mathcal{A}_j \in \mathscr{B}(X_j)$*;*

(iii) *there exists* $n \in \mathbb{N}$ *s.t.* $T_n(t) := (\mathcal{A}S_{\mathcal{B}}(t))^{(*n)}$ *satisfies* $\|T_n(t)\|_{\mathscr{B}(X,X_m)} \le Ce^{at}$.

Then the following localization of the principal part of the spectrum

(1) *there are some distinct complex numbers* $\xi_1, ..., \xi_k \in \Delta_a$, $k \in \mathbb{N}$ *(with the convention* $\{\xi_1, ...\xi_k\} = \emptyset$ *if* $k = 0$*) such that one has*

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, ..., \xi_k\} \subset \Sigma_d(\Lambda).$$

implies the following quantitative growth estimate on the semigroup

(2) *for any* $a' \in (a, \infty) \setminus \{\Re e\, \xi_j, j = 1, ..., k\}$*, there exists some constructive constant* $C_{a'} > 0$ *such that*

$$\forall t \ge 0, \quad \left\| S_\Lambda(t) - \sum_{j=1}^{k} e^{t\Lambda \Pi_{\Lambda,\xi_j}} \Pi_{\Lambda,\xi_j} \right\|_{\mathscr{B}(X)} \le C_{a'} e^{a't}.$$

In particular, the following partial (but principal) spectral mapping theorem holds

$$\forall\, t \ge 0, \quad \forall\, a' > a, \quad \Sigma(e^{\Lambda t}) \cap \Delta_{e^{a't}} = e^{\Sigma(\Lambda) \cap \Delta_{a'} t}.$$

Proof. We have the following representation formula (see for instance the proof of [51, Theorem 2.13]):

$$S_\Lambda(t) f = \sum_{j=1}^{k} S_{\Lambda,\xi_j}(t) f + \sum_{\ell=0}^{n+1} (-1)^\ell S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t) f + \mathcal{Z}(t) f,$$

for any $f \in \mathrm{D}(\Lambda)$ and $t \ge 0$, where

$$\mathcal{Z}(t) f := \lim_{M \to \infty} \frac{(-1)^n}{2i\pi} \int_{a'-iM}^{a'+iM} e^{zt}\, \mathcal{R}_\Lambda(z)\, (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{n+2} f\, dz.$$

On the one hand, we know from **(i)** and **(ii)** that

$$\forall\, \ell = 0, ..., n+1, \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)\|_{\mathscr{B}(X)} \le C_{a'} e^{a't}.$$

On the other hand, because of **(iii)**, we have

$$\sup_{z \in a' + i\mathbb{R}} \|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^n(z)\|_{\mathscr{B}(X,\mathrm{D}(\Lambda^2))} \le K^1_{a'}$$

and because of **(1)**, since Λ generates a semigroup,

$$\sup_{z \in a' + i\mathbb{R}} \|\mathcal{R}_\Lambda(z)\|_{\mathscr{B}(X)} \le K^2_{a'}.$$

Then, we are going to use the resolvent identity

$$\forall z \notin \Sigma(\mathcal{B}), \quad \mathcal{R}_{\mathcal{B}}(z) = z^{-1}[\mathcal{R}_{\mathcal{B}}(z)\mathcal{B} - I] \tag{4.40}$$

to get an estimate on $\|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^2(z)\|_{\mathscr{B}(\mathrm{D}(\Lambda^2),X)}$ if $|z| \geq 1$. Using twice (4.40), we obtain

$$\forall z \in \mathbb{C},\, |z| \geq 1, \quad \|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^2(z)f\|_X \leq K_{a'}^3 |z|^{-2} \|f\|_{\mathrm{D}(\mathcal{B}^2)}$$

and we notice that $\mathrm{D}(\mathcal{B}^2) = \mathrm{D}(\Lambda^2)$ because $\mathcal{A}$ is bounded. We finally obtain

$$\forall z \in \mathbb{C},\, |z| \geq 1, \quad \|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^2(z)f\|_X \leq K_{a'}^3 \frac{1}{1+|z|^2} \|f\|_{\mathrm{D}(\Lambda^2)}.$$

Moreover, we also have

$$\forall z \in \mathbb{C},\, |z| \leq 1, \quad \|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^2(z)f\|_X \leq K_{a'}^4 \frac{1}{1+|z|^2} \|f\|_{\mathrm{D}(\Lambda^2)}.$$

All together, we deduce that

$$\|\mathcal{Z}(t)\|_{\mathscr{B}(X)} \leq K_{a'} \frac{e^{a't}}{2\pi} \int_{\mathbb{R}} \frac{dy}{1+y^2},$$

which yields the result. □

We can now prove the estimate on the semigroup decay (4.34). We apply Proposition 4.2.88 with $a := \max(-\alpha, -\alpha_0) < 0$. We have $\mathcal{E}_1 \subset \mathrm{D}(\mathcal{L}_\lambda^2) \subset \mathcal{E}_0 \subset \mathcal{E}_{-1}$. Assumptions **(i)**, **(ii)** and **(iii)** are nothing but Lemmas 4.2.76, 4.2.77 and 4.2.80. And **(1)** is given by the previous steps of the proof. We hence conclude that we have the decay result (4.34) for any $\alpha' \in (0, \min(\alpha, \alpha_0)) \setminus \{-\mu_\lambda\}$.

Remark 4.2.89. *Thanks to the first order expansion of μ_λ (4.33), we deduce that $\mu_\lambda < 0$ for λ close enough to 0. As a consequence, for any $\alpha_\lambda \in (0, -\mu_\lambda)$, we have*

$$\|e^{\mathcal{L}_\lambda t} - e^{\mathcal{L}_\lambda t}\Pi_{\mathcal{L}_\lambda,0}\|_{\mathscr{B}(\mathcal{E}_0)} \leq C e^{-\alpha_\lambda t}. \tag{4.41}$$

4.2.8 A dissipative Banach norm for the full linearized operator

Let us define a new norm on $\mathcal{E}_0$ by

$$|||h|||_{\mathcal{E}_0} := \eta \|h\|_{\mathcal{E}_0} + \int_0^{+\infty} \|S_{\mathcal{L}_\lambda}(\tau)h\|_{\mathcal{E}_0}\, d\tau, \quad \eta > 0, \tag{4.42}$$

which is well-defined if $\Pi_{\mathcal{L}_\lambda,0} h = 0$ thanks to the estimate (4.41).

Proposition 4.2.90. *There exist $\eta > 0$ and $\alpha_1 > 0$ such that for any $h_{in} \in \mathcal{E}_0$, $\Pi_{\mathcal{L}_\lambda,0} h_{in} = 0$, the solution $h(t) := S_{\mathcal{L}_\lambda}(t)h_{in}$ to the initial value problem (4.14) satisfies:*

$$\forall t \geq 0, \quad \frac{d}{dt}|||h_t|||_{\mathcal{E}_0} \leq -\alpha_1 |||h_t|||_{\mathcal{E}_0^1},$$

where $\mathcal{E}_0^1 := W_x^{s,1} W_v^{2,1}(\langle v\rangle^2 m)$ and $|||\cdot|||_{\mathcal{E}_0^1}$ is defined as in (4.42).

Proof. From the decay property of $\mathcal{L}_\lambda$ provided by (4.41), if $\Pi_{\mathcal{L}_\lambda,0} h = 0$, we have

$$\|S_{\mathcal{L}_\lambda}(\tau) h\|_{\mathcal{E}_0} \le C e^{-\alpha_\lambda \tau} \|h\|_{\mathcal{E}_0}.$$

We thus deduce that the norms $\|\cdot\|_{\mathcal{E}_0}$ and $|||\cdot|||_{\mathcal{E}_0}$ are equivalent for any $\eta > 0$.

Let us now compute the time derivative of the norm $\mathcal{E}_0$ along h_t where h_t solves the linear evolution problem (4.14). Observe that $\Pi_{\mathcal{L}_\lambda,0} h_t = 0$ due to the mass and momentum conservation properties of the linearized equation. Since the x-derivatives commute with the equation, we can set $s = 0$. We first treat the case $L^1_x L^1_v(\langle v\rangle m)$. We compute

$$\frac{d}{dt}|||h_t|||_{\mathcal{E}_0} = \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} \mathcal{L}_\lambda(h_t) \operatorname{sign}(h_t)\, dx \right) \langle v\rangle m \, dv + \int_0^\infty \frac{\partial}{\partial t} \|h_{t+\tau}\|_{\mathcal{E}_0} \, d\tau =: I_1 + I_2.$$

Concerning the first term, arguing as in the proof of Lemma 4.2.76, we have from the dissipativity of $\mathcal{B}_\lambda$ and the bounds on $\mathcal{A}$

$$I_1 \le \eta \left(C \|h_t\|_{\mathcal{E}_0} - K \|h_t\|_{\mathcal{E}_0^1} \right)$$

for some constants $C, K > 0$.

The second term is computed exactly:

$$I_2 = \int_0^\infty \frac{\partial}{\partial t} \|h_{t+\tau}\|_{\mathcal{E}_0} \, d\tau = \int_0^\infty \frac{\partial}{\partial \tau} \|h_{t+\tau}\|_{\mathcal{E}_0} \, d\tau = -\|h_t\|_{\mathcal{E}_0}.$$

The combination of the two last equations yields the desired result by choosing η small enough. The case of higher-order v-derivativeq is treated similarly as in Lemma 4.2.76. □

4.3 The nonlinear Boltzmann equation

4.3.1 The bilinear estimates

Let us recall a bilinear estimate on the nonlinear term in equation (4.1).

Lemma 4.3.91. *In the space $\mathcal{E}^q := W^{\sigma,1}_v W^{s,1}_x(\langle v\rangle^q m)$ with s, $\sigma \in \mathbb{N}$, $s > 6$ and $q \in \mathbb{N}$, the collision operator Q satisfies*

$$\|Q_{e_\lambda}(g, f)\|_{\mathcal{E}^q} \le C(\|g\|_{\mathcal{E}^{q+1}} \|f\|_{\mathcal{E}^q} + \|g\|_{\mathcal{E}^q} \|f\|_{\mathcal{E}^{q+1}})$$

for some constant $C > 0$, where $\mathcal{E}^{q+1}$ is defined as $\mathcal{E}^q$.

The proof is similar to the one done in [51, Lemma 5.16]. We shall only mention the main steps.

Proof. Let us first consider the velocity aspect only of the norm with $\sigma = 0$. Concerning the case of a constant coefficient of inelasticity, we use that the elastic collision operator Q_1 satisfies (cf [75])

$$\|Q_1(g,f)\|_{L^1_v(m)} \leq C(\|f\|_{L^1_v(m)}\|g\|_{L^1_v(\langle v\rangle m)} + \|f\|_{L^1_v(\langle v\rangle m)}\|g\|_{L^1_v(m)}).$$

First, it can be straightforwardly adapted to the case $L^1(\langle v\rangle^q m)$. Then, if v'_λ and v'_0 denotes the post-collisional velocities in the inelastic case and in the elastic case with obvious notations, using the fact that we both have

$$|v'_\lambda|^2 \leq |v|^2 + |v_*|^2$$

and

$$|v'_0|^2 \leq |v|^2 + |v_*|^2,$$

the same proof can be done in the inelastic case. We hence obtain that

$$\|Q_{e_\lambda}(g,f)\|_{L^1_v(\langle v\rangle^q m)} \leq C\left(\|f\|_{L^1_v(\langle v\rangle^q m)}\|g\|_{L^1_v(\langle v\rangle^{q+1} m)} + \|f\|_{L^1_v(\langle v\rangle^{q+1} m)}\|g\|_{L^1_v(\langle v\rangle^q m)}\right). \tag{4.43}$$

Then, from property (4.17) and inequality (4.43), we deduce that

$$\begin{aligned}\|Q_{e_\lambda}(g,f)\|_{W^{\sigma,1}_v(\langle v\rangle^q m)} \quad \leq \quad & C\Big(\|f\|_{W^{\sigma,1}_v(\langle v\rangle^q m)}\|g\|_{W^{\sigma,1}_v(\langle v\rangle^{q+1} m)} + \\ & \|f\|_{W^{\sigma,1}_v(\langle v\rangle^{q+1} m)}\|g\|_{W^{\sigma,1}_v(\langle v\rangle^q m)}\Big)\end{aligned}$$

as well as similar results from the other estimates.

As a final step, we consider the x aspect of the norm. We use the Sobolev embedding $W^{s/2,1}_x(\mathbb{T}^3) \subset L^\infty_x(\mathbb{T}^3)$ with continuous embedding since $s > 6$ and we conclude as in [51]. □

4.3.2 The main results

Let us now give some results on the stability and relaxation to equilibrium for solutions to the full non-linear problem. We consider first the close-to-equilibrium regime (Theorem 4.3.92), and then the weakly inhomogeneous regime (Theorem 4.3.93).

Theorem 4.3.92 (Perturbative solutions close to equilibrium)**.** *Let us consider $\lambda \in [0,\lambda']$ (where λ' is given by Theorem 4.2.82). There is some constructive constant $\varepsilon > 0$ such that for any initial data $f_{in} \in \mathcal{E}_0$ satisfying*

$$\|f_{in} - G_\lambda\|_{\mathcal{E}_0} \leq \varepsilon,$$

and f_{in} has the same global mass and momentum as the equilibrium G_λ defined in subsection 3.1, there exists a unique global solution $f \in L^\infty_t(\mathcal{E}_0)$ to (4.1).

This solution furthermore satisfies that for any $\widetilde{\alpha} \in (0, -\mu_\lambda)$:

$$\forall\, t \geq 0, \quad \|f_t - G_\lambda\|_{\mathcal{E}_0} \leq C e^{-\widetilde{\alpha} t}\|f_{in} - G_\lambda\|_{\mathcal{E}_0}$$

for some constructive constant $C > 0$.

For the following theorem, we only consider the case of a constant restitution coefficient, namely $e_\lambda(\cdot)$ is constant equal to $1-\lambda$, Theorem 4.3.93 is thus a result on equation (4.2).

Theorem 4.3.93 (Weakly inhomogeneous solutions)**.** *Let us consider λ in $[0,\lambda']$. Consider a spatially homogeneous distribution $g_{in} = g_{in}(v) \in W^{2,1}_v\left(\langle v\rangle^5 e^{b\langle v\rangle^\beta}\right)$ with the same global mass and momentum as G_λ.*

There is some constructive constant $\varepsilon(g_{in}) > 0$ such that for any initial data $f_{in} \in \mathcal{E}_0$ satisfying

$$\|f_{in} - g_{in}\|_{\mathcal{E}_0} \le \varepsilon(g_{in}),$$

and f_{in} has the same mass and momentum as G_λ and g_{in}, there exists a unique global solution $f \in L^\infty_t(\mathcal{E}_0)$ to (4.1).

Moreover, this solution satisfies

$$\forall\, t \ge 0, \quad \|f_t - g_t\|_{\mathcal{E}_0} \le C\,\varepsilon(g_{in})$$

and for any $\widetilde{\alpha} \in (0, -\mu_\lambda)$,

$$\forall\, t \ge 0, \quad \|f_t - G_\lambda\|_{\mathcal{E}_0} \le C e^{-\widetilde{\alpha} t}$$

for some constructive constant $C > 0$.

4.3.3 Proof of the main results

Proof of Theorem 4.3.92

The strategy is similar to the one from [51] and we will only mention the main ideas of the proof. We begin by giving the key a priori estimate.

Lemma 4.3.94. *With the notations of Theorem 4.3.92, in the space $\mathcal{E}_0$, a solution f_t to the Boltzmann equation formally writes $f_t = G_\lambda + h_t$, $\Pi_{\mathcal{L}_\lambda,0} h_t = 0$, and h_t satisfies the estimate*

$$\frac{d}{dt}|||h_t|||_{\mathcal{E}_0} \le (C|||h_t|||_{\mathcal{E}_0} - K)|||h_t|||_{\mathcal{E}^1_0}$$

for some constants C, $K > 0$ and with $\mathcal{E}^1_0 := W^{s,1}_x W^{2,1}_v(\langle v\rangle^2 m)$.

Proof. We consider the case $L^1_x L^1_v(\langle v\rangle m)$, we will skip the proof of other cases which is similar. We have

$$\frac{d}{dt}|||h_t|||_{L^1_x L^1_v(\langle v\rangle m)} = I_1 + I_2$$

with

$$\begin{aligned} I_1 := &\, \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} \mathcal{L}_\lambda h_t \operatorname{sign}(h_t)\, dx\right) \langle v\rangle m\, dv \\ &+ \int_0^\infty \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} e^{\tau \mathcal{L}_\lambda}(\mathcal{L}_\lambda h_t) \operatorname{sign}(e^{\tau\mathcal{L}_\lambda} h_t)\, dx\right) \langle v\rangle m\, dv\, d\tau \end{aligned}$$

and

$$
\begin{aligned}
I_2 := &\eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} Q_{e_\lambda}(h_t,h_t)\,\mathrm{sign}(h_t)\,dx \right) \langle v \rangle m\,dv \\
&+ \int_0^\infty \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} e^{\tau \mathcal{L}_\lambda} Q_{e_\lambda}(h_t,h_t)\,\mathrm{sign}(e^{\tau\mathcal{L}_\lambda} h_t)\,dx \right) \langle v \rangle m\,dv\,d\tau
\end{aligned}
$$

We already know from Proposition 4.2.90 that by choosing η small enough, we have

$$
I_1 \leq -K |\!|\!| h_t |\!|\!|_{L^1_x L^1_v(\langle v \rangle^2 m)}, \quad K > 0.
$$

For the second term, we have

$$
\begin{aligned}
I_2 \leq &\eta \int_{\mathbb{R}^3} \|Q_{e_\lambda}(h_t,h_t)\|_{L^1_x(\langle v \rangle m)}\,dv + \int_0^\infty \int_{\mathbb{R}^3} \|e^{\tau\mathcal{L}_\lambda} Q_{e_\lambda}(h_t,h_t)\|_{L^1_x(\langle v \rangle m)}\,dv\,d\tau \\
\leq &\eta\, \|Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v \rangle m)} + \int_0^\infty \|e^{\tau\mathcal{L}_\lambda} Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v \rangle m)}\,d\tau.
\end{aligned}
$$

We thus deduce

$$
\frac{d}{dt} |\!|\!| h_t |\!|\!|_{L^1_x L^1_v(\langle v \rangle m)} \leq -K |\!|\!| h_t |\!|\!|_{L^1_x L^1_v(\langle v \rangle^2 m)} + |\!|\!| Q_{e_\lambda}(h_t,h_t) |\!|\!|_{L^1_x L^1_v(\langle v \rangle m)}.
$$

Now using the bilinear estimate coming from Lemma 4.3.91, the semigroup decay (4.41) and the fact that $\Pi_{\mathcal{L}_\lambda,0} Q_{e_\lambda}(h_t,h_t) = 0$, we obtain

$$
\begin{aligned}
|\!|\!| Q_{e_\lambda}(h_t,h_t) |\!|\!|_{L^1_x L^1_v(\langle v \rangle m)} \leq &\eta\, \|Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v \rangle m)} \\
&+ \int_0^\infty \|S_{\mathcal{L}_\lambda}(\tau) Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v \rangle m)}\,d\tau \\
\leq &\eta\, \|h_t\|_{L^1_x L^1_v(\langle v \rangle m)} \|h_t\|_{L^1_x L^1_v(\langle v \rangle^2 m)} \\
&+ C \left(\int_0^\infty e^{-\alpha_\lambda \tau}\,d\tau \right) \|h_t\|_{L^1_x L^1_v(\langle v \rangle m)} \|h_t\|_{L^1_x L^1_v(\langle v \rangle^2 m)} \\
\leq &C\, \|h_t\|_{L^1_x L^1_v(\langle v \rangle m)} \|h_t\|_{L^1_x L^1_v(\langle v \rangle^2 m)} \\
\leq &C\, |\!|\!| h_t |\!|\!|_{L^1_x L^1_v(\langle v \rangle m)} |\!|\!| h_t |\!|\!|_{L^1_x L^1_v(\langle v \rangle^2 m)},
\end{aligned}
$$

which concludes the proof. □

We shall now construct solutions by considering the following iterative scheme

$$
\partial_t h^{n+1} = \mathcal{L}_\lambda h^{n+1} + Q_{e_\lambda}(h^n,h^n), \quad n \geq 1,
$$

with the initialization

$$
\partial_t h^0 = \mathcal{L}_\lambda h^0, \quad h^0_{in} = h_{in}
$$

and we assume $|\!|\!| h_{in} |\!|\!|_{\mathcal{E}_0} \leq \varepsilon/2$. The functions h^n, $n \geq 0$ are well-defined in $\mathcal{E}_0$ thanks to Theorem 4.2.82.

The proof is split into three steps.

Step 1. Stability of the scheme. Let us prove by induction the following control

$$\forall\, n \geq 0, \quad \sup_{t\geq 0}\left(|||h^n_t|||_{\mathcal{E}_0} + K\int_0^t |||h^n_\tau|||_{\mathcal{E}^1_0}\, d\tau\right) \leq \varepsilon \tag{4.44}$$

as soon as $\varepsilon \leq K/(2C)$.

The initialization is deduced from Proposition 4.2.90 and the fact that $\|h_{in}\|_{\mathcal{E}_0} \leq \varepsilon/2$:

$$\sup_{t\geq 0}\left(|||h^0_t|||_{\mathcal{E}_0} + K\int_0^t |||h^0_\tau|||_{\mathcal{E}^1_0}\, d\tau\right) \leq \varepsilon.$$

Let us now assume that (4.44) is satisfied for any $0 \leq n \leq N \in \mathbb{N}^*$ and let us prove it for $n = N+1$. A similar computation as in Lemma 4.3.94 yields

$$\frac{d}{dt}|||h^{N+1}|||_{\mathcal{E}_0} + K\|h^{N+1}\|_{\mathcal{E}^1_0}\, d\tau \leq C|||Q_{e_\lambda}(h^N, h^N)|||_{\mathcal{E}_0}$$

for some constants $C, K > 0$, which implies

$$\begin{aligned}
\|h^{N+1}_t\|_{\mathcal{E}_0} + K\int_0^t \|h^{N+1}_\tau\|_{\mathcal{E}^1_0}\, d\tau &\leq |||h_{in}|||_{\mathcal{E}_0} + \int_0^t |||Q_{e_\lambda}(h^N_\tau, h^N_\tau)|||_{\mathcal{E}_0}\, d\tau \\
&\leq |||h_{in}|||_{\mathcal{E}_0} + C\left(\sup_{\tau\geq 0}|||h^N_\tau|||_{\mathcal{E}_0}\right)\int_0^t |||h^N_\tau|||_{\mathcal{E}^1_0}\, d\tau \\
&\leq \frac{\varepsilon}{2} + \frac{C}{K}\varepsilon^2 \\
&\leq \varepsilon,
\end{aligned}$$

as soon as $\varepsilon < K/(2C)$.

Step 2. Convergence of the scheme. We denote $d^n := h^{n+1} - h^n$ and $s^n := h^{n+1} + h^n$ for $n \geq 0$. They satisfy

$$\forall\, n \geq 0, \quad \partial_t d^{n+1} = \mathcal{L}_\lambda d^{n+1} + Q_{e_\lambda}(d^n, s^n) + Q_{e_\lambda}(s^n, d^n)$$

and

$$\partial_t d^0 = \mathcal{L}_\lambda d^0 + Q_{e_\lambda}(h^0, h^0).$$

Let us denote

$$A^n(t) := \sup_{0\leq r\leq t}\left(|||d^n_r|||_{\mathcal{E}_0} + K\int_0^r \|d^n_\tau\|_{\mathcal{E}^1_0}\, d\tau\right).$$

We can prove by induction that

$$\forall\, t \geq 0, \quad \forall\, n \geq 0, \quad A^n(t) \leq (\overline{C}\varepsilon)^{n+2}$$

for some constant $\overline{C} > 0$.

Hence for ε small enough, the series $\sum_{n\geq 0} A^n(t)$ is summable for any $t \geq 0$ and the sequence h^n has the Cauchy property in $L^\infty_t(\mathcal{E}_0)$, which proves the convergence of the iterative scheme. The limit h as n goes to infinity satisfies the equation in the strong sense in $\mathcal{E}_0$.

Step 3. Rate of decay. We now consider the solution h constructed so far. From the first step, we first deduce by letting n go to infinity in the stability estimate that

$$\sup_{t\geq 0}\left(|||h_t|||_{\mathcal{E}_0}+K\int_0^t |||h_\tau|||_{\mathcal{E}_0^1}\,d\tau\right)\leq \varepsilon.$$

Second, we can apply the a priori estimate from Lemma 4.3.94 to this solution h which implies that

$$|||h_t|||_{\mathcal{E}_0}\leq e^{-\frac{K}{2}t}\|h_{in}\|_{\mathcal{E}_0}$$

under the appropriate smallness condition on ε. Using the fact that $|||h_t|||_{\mathcal{E}_0}$ converges to zero as $t\to+\infty$, we obtain

$$\int_t^\infty \|h_t\|_{\mathcal{E}_0^1}\,d\tau\leq \frac{2}{K\eta}\|h_t\|_{\mathcal{E}_0}\leq Ce^{-\frac{K}{2}t}\|h_{in}\|_{\mathcal{E}_0}.$$

We shall now perform a bootstrap argument in order to ensure that the solution h_t enjoys the same decay rate $O(e^{-\alpha' t})$ as the linearized semigroup (Theorem 4.2.82). Assuming that the solution is known to decay as

$$\|h_t\|_{\mathcal{E}_0}\leq Ce^{-\alpha_0 t}$$

for some constant $C>0$, we can prove that it indeed decays

$$\|h_t\|_{\mathcal{E}_0}\leq C'e^{-\alpha_1 t}$$

with $\alpha_1=\min(\alpha_0+K/4,\alpha)$. It can be proved using Theorem 4.2.82 and Lemma 4.3.91. Hence, in a finite number of steps, it proves the desired decay rate $O(e^{-\alpha' t})$.

Proof of Theorem 4.3.93

We split the proof into three steps. We will only deal with the case $L^1_xL^1_v(\langle v\rangle m)$.

Step 1. The spatially homogeneous evolution. We consider the spatially homogeneous initial data g_{in}. From [67, Corollary 6.3], we know that it gives rise to a spatially homogeneous solution $g_t\in L^1_v(\langle v\rangle m)$ which satisfies

$$\|g_t-G_\lambda\|_{L^1_v(\langle v\rangle m)}\to 0$$

with explicit exponential rate and $g_t\in L^\infty_t(L^1_v(\langle v\rangle m))\cap L^1_t(L^1_v(\langle v\rangle^2 m))$.

Step 2. Local in time stability estimate. The goal is to construct a solution f_t close to some spatially homogeneous solution g_t which is uniformly bounded in $L^1_xL^1_v(\langle v\rangle m)$. We consider the difference $d_t:=f_t-g_t$ and we write its evolution equation:

$$\begin{aligned}\partial_t d+v\cdot\nabla_x d&=Q_{e_\lambda}(d,d)+Q^+_{e_\lambda}(g,d)+Q^+_{e_\lambda}(d,g)-Q^-_{e_\lambda}(g,d)-Q^-_{e_\lambda}(d,g)+\lambda^\gamma\Delta_v d\\&=\mathbf{P}(d)+\lambda^\gamma\Delta_v d,\end{aligned}$$

where $\mathbf{P}(d) := Q_{e_\lambda}(d,d) + Q^+_{e_\lambda}(g,d) + Q^+_{e_\lambda}(d,g) - Q^-_{e_\lambda}(g,d) - Q^-_{e_\lambda}(d,g)$. We then estimate the time evolution of the $L^1_x L^1_v(\langle v\rangle m)$ norm:

$$\begin{aligned}
&\frac{d}{dt}\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} = \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} (\mathbf{P}(d_t) + \lambda^\gamma \Delta_v d_t)\,\mathrm{sign}\, d_t \, dx\, \langle v\rangle m\, dv \\
&\le C\,\|Q_{e_\lambda}(d_t,d_t)\|_{L^1_x L^1_v(\langle v\rangle m)} + C\,\|Q^+_{e_\lambda}(g_t,d_t)\|_{L^1_x L^1_v(\langle v\rangle m)} + C\,\|Q^+_{e_\lambda}(d_t,g_t)\|_{L^1_x L^1_v(\langle v\rangle m)} \\
&\quad + C\,\|Q^-_{e_\lambda}(d_t,g_t)\|_{L^1_x L^1_v(\langle v\rangle m)} - \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} Q^-_{e_\lambda}(g_t,d_t)\,\mathrm{sign}\, d_t\, dx\, \langle v\rangle m\, dv \\
&\quad + \lambda^\gamma \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v |d_t|\, dx\, \langle v\rangle m\, dv.
\end{aligned}$$

First, using the bilinear estimates of Lemma 4.3.91, we have

$$\|Q_{e_\lambda}(d,d)\|_{L^1_x L^1_v(\langle v\rangle m)} \le C\|d\|_{L^1_x L^1_v(\langle v\rangle m)}\|d\|_{L^1_x L^1_v(\langle v\rangle^2 m)}$$

and

$$\begin{aligned}
\|Q^+_{e_\lambda}(d,g)\|_{L^1_x L^1_v(\langle v\rangle m)} + \|Q^+_{e_\lambda}(g,d)\|_{L^1_x L^1_v(\langle v\rangle m)} &\le \eta\,\|g\|_{L^1_x L^1_v(\langle v\rangle^2 m)}\|d\|_{L^1_x L^1_v(\langle v\rangle^2 m)} \\
&\quad + C_\eta\,\|g\|_{L^1_x L^1_v(\langle v\rangle m)}\|d\|_{L^1_x L^1_v(\langle v\rangle m)}
\end{aligned}$$

for any $\eta > 0$ as small as wanted, and some corresponding η-dependent constant C_η. Second, by trivial explicit computations we have

$$\|Q^-_{e_\lambda}(d,g)\|_{L^1_x L^1_v(\langle v\rangle m)} \le C\,\|d\|_{L^1_x L^1_v(\langle v\rangle m)}\|g\|_{L^1_x L^1_v(\langle v\rangle^2 m)}.$$

Third, we have for some $K > 0$,

$$-\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} Q^-_{e_\lambda}(g,d)\,\mathrm{sign}\, d_t\, dx\, \langle v\rangle m\, dv \le -K\|d\|_{L^1_x L^1_v(\langle v\rangle^2 m)}.$$

Fourth and last,

$$\lambda^\gamma \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v |d|\, dx\, \langle v\rangle m\, dv \le C\,\|d\|_{L^1_x L^1_v(\langle v\rangle m)} \le C\,\|d\|_{L^1_x L^1_v(\langle v\rangle^2 m)}.$$

Gathering all these estimates, we finally obtain

$$\begin{aligned}
\frac{d}{dt}\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} &\le (C\,\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} + \lambda^\gamma - K)\|d_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)} \\
&\quad + C\,\|g_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)}\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)}.
\end{aligned}$$

We then introduce an iterative scheme

$$\partial_t d^{n+1} = Q_{e_\lambda}(d^n, d^n) + Q_{e_\lambda}(g, d^n) + Q_{e_\lambda}(d^n, g), \quad n \ge 0,$$

and

$$\partial_t d^0 = Q_{e_\lambda}(g, d^0) + Q_{e_\lambda}(d^0, g)$$

with $d_{in}^n = d_{in} = f_{in} - g_{in}$ for all $n \geq 0$, just as the previous subsection. At each step, a global solution d_n is constructed in $L^1_x L^1_v(\langle v\rangle m)$ using the estimates above. We assume that $\|d_{in}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon/2$. By passing to the limit in the a priori estimates, we deduce that, as long as

$$C\,\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq K - \lambda^\gamma \tag{4.45}$$

we have

$$\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\varepsilon}{2} \exp\left(C \int_0^t \|g_\tau\|_{L^1_x L^1_v(\langle v\rangle^2 m)}\, d\tau\right).$$

We then choose ε small enough so that $C\varepsilon \leq K - \lambda^\gamma$, and then since $g_t \in L^1_t\left(L^1_x L^1_v(\langle v\rangle^2 m)\right)$, we can choose $T_1 = T_1(\varepsilon) > 0$ so that the smallness condition (4.45) is satisfied and

$$\forall\, t \in [0, T_1], \quad \|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon.$$

Observe that $T_1(\varepsilon) \xrightarrow[\varepsilon \to 0]{} +\infty$. This completes the proof of stability.

Step 3. The trapping mechanism. Consider δ the smallness constant of the stability neighborhood in Theorem 4.3.92 in $L^1_x L^1_v(\langle v\rangle m)$. Then from [67], we deduce that there is some time $T_2 = T_2(M) > 0$ such that

$$\forall\, t \geq T_2, \quad \|g_t - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\delta}{3}$$

where $G_{\lambda,g}$ is the equilibrium associated to g_{in}. We then choose ε small enough such that

$$\|f_{in} - g_{in}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon \Rightarrow \|G_{\lambda,f} - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\delta}{3}$$

where $G_{\lambda,f}$ is the equilibrium associated to f_{in}, $T_1(\varepsilon) \geq T_2(M)$ and

$$\|f_{T_2} - g_{T_2}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\delta}{3},$$

from the stability result.

We deduce that

$$\begin{aligned}\|f_{T_2} - G_{\lambda,f}\|_{L^1_x L^1_v(\langle v\rangle m)} &\leq \|f_{T_2} - g_{T_2}\|_{L^1_x L^1_v(\langle v\rangle m)} + \|g_{T_2} - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \\ &\quad + \|G_{\lambda,f} - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \\ &\leq \delta\end{aligned}$$

and we can therefore use the perturbative Theorem 4.3.92 for $t \geq T_2$ which concludes the proof.

Remark 4.3.95. *In the case of a non-constant coefficient of inelasticity, we can prove such a result in a weakly inhomogeneous setting considering an homogeneous distribution $g_{in} = g_{in}(v)$ which is close enough to the equilibrium. Indeed, using Theorem 4.3.92, we obtain the existence of a solution of the equation (4.1) which converges to the equilibrium. However, we can not conclude if we do not suppose that g_{in} is close enough to G_λ.*

Troisième partie

Equations de Fokker-Planck

Chapitre 5

Uniform semigroup spectral analysis of the discret, fractional and classical Fokker-Planck equations

RÉSUMÉ. Dans cette partie, nous nous intéressons à l'analyse spectrale des équations de Fokker-Planck discrète, fractionnaire et classique du point de vue des semi-groupes. Les équations de Fokker-Planck discrète et fractionnaire convergent en un certain sens vers l'équation de Fokker-Planck classique. Nous traitons donc dans un premier temps les équations de Fokker-Planck discrète et classique dans un même cadre, prouvant des estimations spectrales uniformes grâce à un argument perturbatif. Et dans un second temps, nous réalisons une analyse similaire pour les équations de Fokker-Planck fractionnaire et classique grâce à un argument d'élargissement de l'espace dans lequel le semi-groupe décroit.

ABSTRACT. In this part, we investigate the spectral analysis (from the point of view of semigroups) of discrete, fractional and classical Fokker-Planck equations. Discrete and fractional Fokker-Planck equations converge in some sense to the classical one. As a consequence, we first deal with discrete and classical Fokker-Planck equations in a same framework, proving uniform spectral estimates using a perturbation argument. Then, we do a similar analysis for fractional and classical Fokker-Planck equations using an enlargement of the space in which the semigroup decays.

5.1 Introduction

In this paper, we investigate from a spectral analysis point of view some discret and fractional Fokker-Planck equations. They are simple models for describing the time evolution of a density function $f = f(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, of particles undergoing both diffusion and (harmonic) confinement mechanisms and write

$$\partial_t f = \Lambda_\varepsilon f = \mathcal{D}_\varepsilon f + \operatorname{div}(xf). \tag{5.1}$$

The diffusion term may be either a discret diffusion

$$\mathcal{D}_\varepsilon(f) := \frac{1}{\varepsilon^2}(k_\varepsilon * f - f),$$

for a convenient (centered, nonnegative, smooth and decaying fast enough) kernel k, with the usual notation $k_\varepsilon(x) = k(x/\varepsilon)/\varepsilon^d$, $\varepsilon > 0$. It can also be a fractional diffusion

$$\mathcal{D}_\varepsilon(f)(x) := c_\varepsilon \int_{\mathbb{R}^d} \frac{f(y) - f(x) - \chi(x-y)(y-x)\cdot\nabla f(x)}{|x-y|^{d+2-\varepsilon}}\, dy, \tag{5.2}$$

with $\varepsilon \in (0, 2)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$, and a convenient renormalization constant $c_\varepsilon > 0$. Both families of equations are related to the classical Fokker-Planck equation, because in the limit $\varepsilon \to 0$, one may recover

$$\partial_t f = \Lambda_0 f = \Delta f + \operatorname{div}(xf).$$

The main features of these equations are (expected to be) the same: they are mass preserving, positivity preserving, have a unique positive stationary state with unit mass and that stationary state is exponentially stable, in particular

$$f(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{5.3}$$

for any solution associated to an initial datum f_0 with vanishing mass. Such results can be obtained using different tools as the spectral analysis of self-adjoint operators, some (generalization of) Poincaré inequalities or logarithmic Sobolev inequalities as well as the Krein-Rutman theory for positive semigroup.

The aim of this paper is to initiate a kind of unified treatment of these equations and more importantly to establish that the convergence (5.3) is exponentially fast uniformly with respect to the diffusion term for a large class of initial data which are taken in a fixed (large) weighted Lebesgue or weighted Sobolev space X. Our approach is a semigroup approach in the spirit of the semigroup decomposition framework introduced by Mouhot in [75] and developed subsequently in [66, 51, 90, 64]. A typical result we are able to prove is the following.

Theorem 5.1.1 (rough version)**.** *There exist $\varepsilon_0 \in (0, 2)$, $a < 0$ and $C \geq 1$ such that:*

$$\|S_{\Lambda_\varepsilon}(t) f - \Pi_{\Lambda_\varepsilon,0} S_{\Lambda_\varepsilon}(t) f\|_X \leq C\, e^{at}\ \ \|f - \Pi_{\Lambda_\varepsilon,0} f\|_X \quad \forall\, t \geq 0,\ \forall\, \varepsilon \in [0, \varepsilon_0],\ \forall\, f \in X,$$

where X is (for instance) a L^1 weighted space, $S_{\Lambda_\varepsilon}(t) = e^{\Lambda_\varepsilon t}$ stands for the semigroup associated to the generator Λ_ε and $\Pi_{\Lambda_\varepsilon,0}$ for the projector onto the null space of Λ_ε.

Theorem 5.1.1 generalizes to the discrete diffusion Fokker-Planck equation similar results obtained for the classical Fokker-Planck equation in [51, 64] and makes uniform with respect to the fractional diffusion parameter the convergence results obtained for the fractional diffusion equation in [90]. It is worth mentioning that there exists a huge literature on the long-time behaviour for the Fokker-Planck equation as well as (to a lesser extend) for the fractional Fokker-Planck equation. We refer to the references quoted in [51, 64, 90] for details. There also probably exist many papers on the discrete diffusion equation since it is strongly related to a standard random walk in $\mathbb{R}^d$, but we were not able to find any precise reference in this PDE context.

Let us explain our method. First, we may associate a semigroup S_{Λ_ε} to the evolution equation (5.1) in many Sobolev spaces, and that semigroup is mass preserving and strongly positive. In other words, S_{Λ_ε} is a Markov semigroup and it is then expected that there exists a unique positive and unit mass steady state G_ε to the equation (5.1). Next, we are able to establish that the semigroup splits as

$$S_{\Lambda_\varepsilon} = S^1_\varepsilon + S^2_\varepsilon, \quad S^1_\varepsilon \approx e^{tT_\varepsilon}, \ T_\varepsilon \text{ finite dimensional}, \quad S^2_\varepsilon = \mathcal{O}(e^{at}), \ a < 0, \tag{5.4}$$

in these many weighted Sobolev spaces. The above decomposition of the semigroup is the main technical issue of the paper. It is obtained by introducing a convenient splitting

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon \tag{5.5}$$

where $\mathcal{B}_\varepsilon$ enjoys suitable dissipativity property and $\mathcal{A}_\varepsilon$ enjoys some suitable $\mathcal{B}_\varepsilon$-power regularity (by analogy with the $\mathcal{B}_\varepsilon$-power compactness notion introduced by Voigt [96]). It is worth emphasizing that we are able to exhibit such a splitting with uniform (dissipativity, regularity) estimates with respect to the diffusion parameter $\varepsilon \in [0, \varepsilon_0]$ in several weighted Sobolev spaces.

As a consequence of (5.4), we may indeed apply the Krein-Rutman theory developed in [70, 63] and exhibit such a unique positive and unit mass steady state G_ε. Of course for the classical and fractional Fokker-Planck equations the steady state is trivially given through an explicit formula (the Krein-Rutman theory is useless in that cases). A next direct consequence of the above spectral and semigroup decomposition (5.4) is that there is a spectral gap in the spectral set $\Sigma(\Lambda_\varepsilon)$ of the generator Λ_ε, namely

$$\lambda_\varepsilon := \sup\{\Re e\, \xi \in \Sigma(\Lambda_\varepsilon) \backslash \{0\}\} < 0, \tag{5.6}$$

and then that an exponential trend to equilibrium can be established, namely

$$\|S_{\Lambda_\varepsilon}(t)\ f_0 - G_\varepsilon\|_X \le C_\varepsilon\, e^{at}\, \|f_0 - G_\varepsilon\|_X \quad \forall\, t \ge 0, \quad \forall\, \varepsilon \in [0, \varepsilon_0], \ \forall\, a > \lambda_\varepsilon, \tag{5.7}$$

for any unit mass initial datum $f_0 \in X$.

Our next step consists in proving that the spectral gap (5.6) and the estimate (5.7) are uniform with respect to ε, more precisely, there exists $\lambda^* < 0$ such that $\lambda_\varepsilon \le \lambda^*$ for any $\varepsilon \in [0, \varepsilon_0]$ and C_ε can be chosen independent to $\varepsilon \in [0, \varepsilon_0]$.

A first way to get such uniform bounds is just to have in at least one Hilbert space $E_\varepsilon \subset L^1(\mathbb{R}^d)$ the estimate

$$\forall f \in \mathcal{D}(\mathbb{R}^d),\ \langle f \rangle = \int_{\mathbb{R}^d} f\,dx = 0, \quad (\Lambda_\varepsilon f, f)_{E_\varepsilon} \le \lambda^* \|f\|^2_{E_\varepsilon},$$

and then (5.7) essentially follows from the fact that the splitting (5.5) is true with operators which are uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$. It is the strategy we use in the case of the fractional diffusion and the work has already been made in [90] except for the simple but fundamental observation that the fractional diffusion operator is uniformly bounded (and converges to the classical diffusion operator) when it is suitable (re)scaled.

A second way to get the desired uniform estimate is to use a perturbation argument. Observing that

$$\forall \varepsilon \in [0, \varepsilon_0], \quad \Lambda_\varepsilon - \Lambda_0 \ = \mathcal{O}(\varepsilon),$$

for a suitable operator norm, we are able to deduce that $\varepsilon \mapsto \lambda_\varepsilon$ is a continuous function at 0, from which we readily conclude. We use here again that the considered model converges to the classical Fokker-Planck equation. In other words, this model can be seen as a (singular) perturbation to the limit equation and our analyze takes advantage of such a property in order to capture the asymptotic behaviour of the related spectral objects (spectrum, spectral projector, ...) in order to get the above uniform spectral decomposition.

Let us present now some comments and possible extensions.

Motivations. The main motivation of the present work is rather theoretical and methodological. Spectral gap and semigroup estimates in large Lebesgue spaces have been established both for Boltzmann like equations and Fokker-Planck like equations in a series of recent papers [75, 66, 51, 70, 40, 27, 90, 64, 69]. The proofs are based on a splitting of the generator method as here and previously explained, but the appropriate splitting are rather different for the two kinds of models. The operator $\mathcal{A}_\varepsilon$ is a multiplication (0-order) operator for a Fokker-Planck equation while it is an integral (-1-order) operator for a Boltzmann equation. More importantly, the fundamental and necessary regularizing effect is given by the action of the semigroup $S_{\mathcal{B}_\varepsilon}$ for the Fokker-Planck equation while it is given by the action of the operator $\mathcal{A}_\varepsilon$ for the Boltzmann equation. Our purpose is precisely to show that all these equations can be handled in the same framework, by exhibiting a suitable and compatible splitting (5.5) which does not blow up and such that the time indexed family of operators $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ (or some iterated convolution products of that one) have a good regularizing property which is uniform in the singular limit $\varepsilon \to 0$.

Probability interpretation. The discret and fractional Fokker-Planck equations are the evolution equations satisfied by the law of the stochastic process which is solution to the SDE

$$dX_t = -X_t dt - d\mathscr{L}_t^\varepsilon,$$

where $\mathscr{L}_t^\varepsilon$ is the Levy (jump) process associated to $k_\varepsilon/\varepsilon^2$ or $c_\varepsilon/|z|^{d+2-\varepsilon}$. For two trajectories X_t and Y_t to the above SDE associated to some initial datum X_0 and Y_0, and $p \in [1,2)$, we have

$$d|X_t - Y_t|^p = -p|X_t - Y_t|^p dt,$$

from which we deduce

$$\mathscr{E}(|X_t - Y_t|^p) \le e^{-pt}\,\mathscr{E}(|X_0 - Y_0|^p), \quad \forall\, t \ge 0.$$

Denoting by $f_\varepsilon(t)$ the law of X_t and G_ε the law of the stable process Y_t, we classically deduce the Wasserstein distance estimate

$$W_p(f_\varepsilon(t), G_\varepsilon) \le e^{-t}\, W_p(f_0, G_\varepsilon), \quad \forall\, t \ge 0. \tag{5.8}$$

Estimate (5.8) has to be compared with (5.7). While the proof of (5.8) is just straightforward, the proof of (5.7) is not. In particular, for $p = 1$, the Kantorovich-Rubinstein Theorem says that (5.8) is equivalent to the estimate

$$\|f_\varepsilon(t) - G_\varepsilon\|_{(W^{1,\infty}(\mathbb{R}^d))'} \le e^{-t}\, \|f_0 - G_\varepsilon\|_{(W^{1,\infty}(\mathbb{R}^d))'}, \quad \forall\, t \ge 0. \tag{5.9}$$

Estimates (5.8) and (5.9) have to be compared with (5.7). Proceeding in a similar way as in [70, 64] it is likely that the spectral gap estimate (5.9) can be extended (by “shrinkage of the space”) to a weighted Lebesgue space framework and then to get the estimate in Theorem 5.1.1 for any $a \in (-1, 0)$.

Trotter-Kato. From the Trotter-Kato formula

$$S_{\Lambda_\varepsilon} - S_{\Lambda_0} = S_{\Lambda_\varepsilon} * (\Lambda_\varepsilon - \Lambda_0) S_{\Lambda_0}$$

and the two observations

$$D(\Lambda_0^{1/4}) \subset D(\Lambda_\varepsilon) \subset D(\Lambda_0), \quad \|\Lambda_\varepsilon - \Lambda_0\|_{D(\Lambda_0^3) \to X} = \mathcal{O}(\varepsilon),$$

we should deduce

$$\|S_{\Lambda_\varepsilon} - S_{\Lambda_0}\|_{D(\Lambda_0^2) \to X} = \mathcal{O}(\varepsilon).$$

We believe that these arguments can be made rigorous and then that the same analysis we have performed here should make possible to improve the above estimate into

$$\sup_{t \ge 0} \|S_{\Lambda_\varepsilon}(t) - S_{\Lambda_0}(t)\|_{\mathscr{B}(X)}\, e^{-at} \; = \mathcal{O}(\varepsilon).$$

From discrete to fractional Fokker-Planck equation. We are also convinced that an analogous study can be carried out to handle in a uniform framework discrete fractional Fokker-Planck equations, similarly (but in a more complicated way) as the discrete and classical Fokker-Planck equations.

Singular kernel and other confinement term. We also believe that a similar analysis can be handle with more singular kernels that the ones considered here, the typical example should be $k(z) = (\delta_{-1}+\delta_1)/2$ in dimension $d = 1$, and for confinement term different from the harmonic confinement considered here, including other forces or discrete confinement term. In order to perform such an analysis one could use some trick developed in [70] in order to handle the equal mitosis (which uses one more iteration of the convolution product of the time indexed family of operators $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$).

Linearized and nonlinear equations. We also believe that a similar analysis can be adapted to nonlinear equations. The typical example we have in mind is the Landau grazing collision limit of the Boltzmann equation. One can then expect to get an exponential trend of solutions to its associated Maxwellian equilibrium which is uniform with respect to the considered model (Boltzmann equation with and without Grad's cutoff and Landau equation).

Kinetic like models. A more challenging issue would be to extend the uniform asymptotic analysis to the Langevin SDE or the kinetic Fokker-Planck equation by using some idea developed in [27] which make possible to connect (from a spectral analysis point of view) the parabolic-parabolic Keller-Segel equation to the parabolic-elliptic Keller-Segel equation. The next step should be to apply the theory to the Navier-Stokes diffusion limit of the (in)elastic Boltzmann equation. These more technical problems will be investigated in next works.

Let us end the introduction by describing the plan of the paper. In Section 5.2, we deal with the discrete and classical Fokker-Planck equations in a uniform framework. Section 5.3 is dedicated to the analysis of the fractional and classical Fokker-Planck equations.

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5.2 From discrete to classical Fokker-Planck equation

For a (measurable) moment function $m : \mathbb{R}^d \to \mathbb{R}_+$, we define the norms

$$\|f\|_{L^p(m)} := \|f\, m\|_{L^p(\mathbb{R}^d)}, \quad \|f\|^p_{W^{k,p}(m)} := \sum_{i=0}^k \|\partial^i f\|^p_{L^p(m)}, \quad k \ge 1,$$

and the associated weighted Lebesgue and Sobolev spaces $L^p(m)$ and $W^{k,p}(m)$, we denote $H^k(m) = W^{k,2}(m)$ for $k \ge 1$. We also use the shorthand L^p_r and $W^{1,p}_r$ for the Lebesgue and Sobolev spaces $L^p(m)$ and $W^{1,p}(m)$ when $m(x) = \langle x \rangle^r$, $\langle x \rangle := (1+|x|^2)^{1/2}$. From now on, we fix a polynomial weight $m(x) := \langle x \rangle^q$ with $q > d/2 + 5$.

In the sequel, we consider a kernel $k \in W^{2,1}(\mathbb{R}^d) \cap L^1_{2q+3}$ satisfying the centered condition

$$\int_{\mathbb{R}^d} k(x) \begin{pmatrix} 1 \\ x \\ x \otimes x \end{pmatrix} dx = \begin{pmatrix} 1 \\ 0 \\ 2I_d \end{pmatrix}, \tag{5.1}$$

as well as the positivity condition: there exist $\kappa, r > 0$ such that

$$k \geq \kappa \, \mathbf{1}_{B(0,r)}. \tag{5.2}$$

Let us notice that assumptions made on k imply

$$\widehat{k}(\xi)^2 \leq C \, \frac{1 - \widehat{k}(\xi)}{|\xi|^2}, \quad \forall \, \xi \in \mathbb{R}^d \tag{5.3}$$

for some constant $C > 0$.

We define $k_\varepsilon(x) := 1/\varepsilon^d k(x/\varepsilon)$, $x \in \mathbb{R}^d$ for $\varepsilon > 0$, and we consider the discrete and classical Fokker-Planck equations

$$\begin{cases} \partial_t f = \dfrac{1}{\varepsilon^2}(k_\varepsilon *_x f - f) + \mathrm{div}_x(xf) =: \Lambda_\varepsilon f, \quad \varepsilon > 0 \\ \partial_t f = \Delta_x f + \mathrm{div}_x(xf) =: \Lambda_0 f. \end{cases} \tag{5.4}$$

The main result of the section reads as follows.

Theorem 5.2.1. *Assume $q > d/2 + 5$ and consider a kernel $k \in W^{2,1}(\mathbb{R}^d) \cap L^1_{2q+3}$ which satisfies* (5.1) *and* (5.2).

(1) For any $\varepsilon > 0$, there exists a positive and unit mass normalized steady state $G_\varepsilon \in L^1_q(\mathbb{R}^d)$ to the discrete Fokker-Planck equation (5.4).

(2) There exist an explicit constant $a_0 < 0$ and a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, the semigroup $S_{\Lambda_\varepsilon}(t)$ associated to the discrete Fokker-Planck equation (5.4) *satisfies : for any $f \in L^1_q$ and any $a > a_0$,*

$$\|S_{\Lambda_\varepsilon}(t) f - G_\varepsilon \langle f \rangle\|_{L^1_q} \leq C_a \, e^{at} \, \|f - G_\varepsilon \langle f \rangle\|_{L^1_q}, \quad \forall \, t \geq 0$$

for some explicit constant $C_a > 0$. In particular, the spectrum $\Sigma(\Lambda_\varepsilon)$ of Λ_ε satisfies the separation property $\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_0} = \{0\}$ in L^1_q.

The method of the proof consists in introducing a suitable splitting $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$, in establishing some dissipativity and regularity properties on $\mathcal{B}_\varepsilon$ and $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ and finally to apply the Krein-Rutman theory revisisted in [70, 63] and the perturbation theory developed in [66, 90, 63].

5.2.1 Splitting of Λ_ε for $\varepsilon \geq 0$

For a given radially symmetric function $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$, we define χ_R by $\chi_R(x) := \chi(x/R)$ for $R > 0$ and we denote $\chi_R^c := 1 - \chi_R$.

We define the splitting of Λ_ε for $\varepsilon \geq 0$ as follows.

Splitting of Λ_ε for $\varepsilon > 0$. We define

$$\mathcal{A}_\varepsilon f := M\, \chi_R\, (k_\varepsilon * f)$$

and

$$\mathcal{B}_\varepsilon f := \left(\frac{1}{\varepsilon^2} - M\right)(k_\varepsilon * f - f) + M\, \chi_R^c\, (k_\varepsilon * f - f) + \operatorname{div}(xf) - M\, \chi_R\, f,$$

for some constants M, R to be chosen later. One can notice that $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$.

Splitting of Λ_0. We define $\mathcal{A}_0 f := M\, \chi_R f$ and $\mathcal{B}_0 f := \Lambda_0 f - M\, \chi_R f$ so that $\Lambda_0 = \mathcal{A}_0 + \mathcal{B}_0$.

5.2.2 Convergences $\mathcal{A}_\varepsilon \to \mathcal{A}_0$ and $\mathcal{B}_\varepsilon \to \mathcal{B}_0$.

Lemma 5.2.2. *Consider $s \in \mathbb{N}$. The following convergences hold:*

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathscr{B}(H^s(m))} \xrightarrow[\varepsilon \to 0]{} 0 \quad \textit{and} \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathscr{B}(H^{s+3}(m), H^s(m))} \xrightarrow[\varepsilon \to 0]{} 0.$$

Proof. Step 1. We first deal with $\mathcal{A}_\varepsilon$ in the case $s = 0$:

$$\|\mathcal{A}_\varepsilon f - \mathcal{A}_0 f\|_{L^2(m)} = \|M\, \chi_R\, (k_\varepsilon * f - f)\, m\|_{L^2} \leq C\, \|k_\varepsilon * f - f\|_{L^2} \xrightarrow[\varepsilon \to 0]{} 0.$$

Concerning the first derivative, writing that

$$\partial_x(\mathcal{A}_\varepsilon f - \mathcal{A}_0 f) = M\, (\partial_x \chi_R)\, (k_\varepsilon * f - f) + M\, \chi_R\, (k_\varepsilon * \partial_x f - \partial_x f)$$

and using that $\partial_x \chi_R$ is uniformly boounded as well as χ_R, we obtain the result. We omit the details of the proof for higher order derivatives.

Step 2. In order to prove the second part of the result, we are going to prove that

$$\|\Lambda_\varepsilon - \Lambda_0\|_{\mathscr{B}(H^{s+3}(m), H^s(m))} \xrightarrow[\varepsilon \to 0]{} 0.$$

First, let us remark that $(\Lambda_\varepsilon - \Lambda_0) f = 1/\varepsilon^2 (k_\varepsilon * f - f) - \Delta f$. Using (5.1), we have

$$\Lambda_\varepsilon f(x) - \Lambda_0 f(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} k_\varepsilon(x-y)(f(y) - f(x))\, dy - \Delta f(x).$$

We now write a Taylor development of f between x and y:

$$\begin{aligned} f(y) - f(x) &= (y-x) \cdot \nabla f(x) + \frac{1}{2}\, D^2 f(x)(y-x, y-x) \\ &\quad + \frac{1}{2} \int_0^1 (1-s)^2 D^3 f(x + s(y-x))(y-x, y-x, y-x)\, ds, \end{aligned}$$

the first involving the gradient of f will give no contribution using (5.1). Performing a change of variables, we obtain:

$$\begin{aligned}&\Lambda_\varepsilon f(x) - \Lambda_0 f(x)\\ &= \int_{\mathbb{R}^d} k(z)\left(\frac{1}{2}D^2 f(x)(z,z) + \frac{\varepsilon}{2}\int_0^1 (1-s)^2 D^3 f(x+s\varepsilon z)(z,z,z)\,ds\right)dz - \Delta f(x).\end{aligned}$$

Using that

$$D^2 f(x)(z,z) = \sum_{i=1}^n z_i^2 \frac{\partial^2 f(x)}{\partial x_i^2} + \sum_{i\neq j} z_i z_j \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \tag{5.5}$$

and (5.1), we deduce that the first term on the right-hand side of (5.5) will be canceled by $\Delta f(x)$ and that the second one vanishes. It thus implies

$$\Lambda_\varepsilon f(x) - \Lambda_0 f(x) = \frac{\varepsilon}{2}\int_{\mathbb{R}^d} k(z)\int_0^1 (1-s)^2 D^3 f(x+s\varepsilon z)(z,z,z)\,ds\,dz.$$

Consequently, using (5.1) and Jensen inequality

$$\begin{aligned}&\|\Lambda_\varepsilon - \Lambda_0\|_{L^2(m)}\\ &\le C\,\varepsilon\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} k(z)\int_0^1 (1-s)^2 D^3 f(x+sz)(z,z,z)\,ds\,dz\right)^2 m^2(x)\,dx\right)^{1/2}\\ &\le C\,\varepsilon\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(z)\,|z|^3\int_0^1 |D^3 f(x+s\varepsilon z)|^2\,m^2(x+s\varepsilon z)\,m^2(s\varepsilon z)\,ds\,dz\,dx\right)^{1/2}\\ &\le C\,\varepsilon\left(\int_{\mathbb{R}^d}|D^3 f(x)|^2 m^2(x)\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d} k(z)\,|z|^3\,m^2(z)\,dz\right)^{1/2}\\ &\le C\,\varepsilon\,\|f\|_{H^3(m)} \xrightarrow[\varepsilon\to 0]{} 0,\end{aligned}$$

where we have used $k \in L^1_{2q+3}$ and this concludes the proof of the second part in the case $s=0$. Since the operator ∂_x commutes with $\Lambda_\varepsilon - \Lambda_0$, there is no need here to write the proof for $s>0$. □

5.2.3 Uniform boundedness of $\mathcal{A}_\varepsilon$

Lemma 5.2.3. *For any $p \in [1,\infty]$, $s \ge 0$ and any weight function $m \ge 1$, the operator $\mathcal{A}_\varepsilon$ is bounded from $W^{s,p}$ into $W^{s,p}(m)$ with a norm which does not depend on ε.*

Proof. For any $f \in L^p(m)$, we have

$$\|\mathcal{A}_\varepsilon f\|_{L^p(m)} \le C\,\|k_\varepsilon * f\|_{L^p} \le C\,\|k_\varepsilon\|_{L^1}\,\|f\|_{L^p}.$$

thanks to the Young inequality. We conclude that $\mathcal{A}_\varepsilon$ is bounded from L^p into $L^p(m)$ by observing that $\|k_\varepsilon\|_{L^1} = \|k\|_{L^1} = 1$. The proof for the case $s>0$ is similar and it is thus skipped. □

5.2.4 Uniform dissipativity properties of $\mathcal{B}_\varepsilon$

Lemma 5.2.4. *We suppose that $q > d/2$. For any $a > d/2 - q$, there exist $\varepsilon_0 > 0$, $M \geq 0$ and $R \geq 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{B}_\varepsilon - a$ is dissipative in $L^2(m)$.*

Proof. We consider $a > d/2 - q$. We are going to estimate the integral $\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f)\, f\, m^2$ for $\varepsilon > 0$ which can be split into several pieces:

$$\begin{aligned}\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f)\, f\, m^2 &= \left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d} (k_\varepsilon * f - f)\, f\, m^2 + \int_{\mathbb{R}^d} M\, \chi_R^c\, (k_\varepsilon * f - f)\, f\, m^2 \\ &\quad + \int_{\mathbb{R}^d} \mathrm{div}(xf)\, f\, m^2 - \int_{\mathbb{R}^d} M\, \chi_R\, f^2\, m^2 \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

We choose $M \geq 0$ and $R \geq 0$ large enough so that

$$\forall\, x \in \mathbb{R}^d, \quad \widetilde{\psi}_m(x) - M\, \chi_R(x) \leq \frac{a}{2} + \frac{1}{2}\left(\frac{d}{2} - q\right)$$

where $\widetilde{\psi}_m$ is defined in (5.11) and is such that $\widetilde{\psi}_m \xrightarrow[|x|\to\infty]{} d/2 - q$. We then fix $\varepsilon_1 > 0$ such that $M \leq 1/(2\varepsilon_1^2)$ and consider $\varepsilon \in (0, \varepsilon_1]$.

We first deal with T_1 performing a classical computation and using that $\int_{\mathbb{R}^d} k_\varepsilon = 1$:

$$\begin{aligned}T_1 &= \left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d\times\mathbb{R}^d} k_\varepsilon(x-y)\, (f(y) - f(x))\, f(x)\, m^2(x)\, dy\, dx \\ &= -\frac{1}{2}\left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d\times\mathbb{R}^d} (f(y) - f(x))^2\, k_\varepsilon(x-y)\, m^2(x)\, dy\, dx \\ &\quad + \frac{1}{2}\left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(f^2(y) - f^2(x)\right)\, k_\varepsilon(x-y)\, m^2(x)\, dy\, dx \\ &\leq \frac{1}{2}\left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(m^2(y) - m^2(x)\right)\, k_\varepsilon(x-y)\, f^2(x)\, dy\, dx\end{aligned}$$

where we have performed a change of variables to get the last inequality. We then write a Taylor development of m^2 between x and y:

$$m^2(y) - m^2(x) = (y-x)\cdot\nabla m^2(x) + \frac{1}{2} D^2 m^2(x + \theta(y-x))(y-x, y-x)$$

for some $\theta \in (0,1)$. The term involving the gradient of m^2 will give no contribution because of (5.1) and using that

$$|D^2 m^2(x + \theta(y-x))(y-x, y-x)| \leq C\, |x-y|^2\, \langle x\rangle^{2q-2}\, \langle x-y\rangle^{2q-2},$$

and that $k \in L^1_{2q}$, we obtain

$$\begin{aligned}T_1 &\leq C\, \left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d\times\mathbb{R}^d} k_\varepsilon(x-y)\, |x-y|^2\, \langle x-y\rangle^{2q-2}\, dy\, f^2(x) \langle x\rangle^{2q-2}\, dx \\ &\leq C\, \left(\frac{1}{\varepsilon^2} - M\right) \varepsilon^2 \int_{\mathbb{R}^d} k(z)\, |z|^2\, \langle z\rangle^{2q-2}\, dz \int_{\mathbb{R}^d} f^2(x)\, \langle x\rangle^{2q-2}\, dx \qquad (5.6) \\ &\leq C \int_{\mathbb{R}^d} f^2(x)\, \langle x\rangle^{2q-2}\, dx.\end{aligned}$$

We now treat the second term T_2:

$$\begin{aligned} T_2 &= \int_{\mathbb{R}^d\times\mathbb{R}^d} M\,\chi_R^c(x)\,k_\varepsilon(x-y)\,f(x)\,f(y)\,m^2(x)\,dy\,dx - \int_{\mathbb{R}^d} M\,\chi_R^c(x)\,f^2(x)\,m^2(x)\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} M\,\chi_R^c(x)\,k_\varepsilon(x-y)\,f^2(x)\,m^2(x)\,dy\,dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} M\,\chi_R^c(x)\,k_\varepsilon(x-y)\,f^2(y)\,m^2(x)\,dy\,dx - \int_{\mathbb{R}^d} M\,\chi_R^c(x)\,f^2(x)\,m^2(x)\,dx \\ &=: T_{21}+T_{22}+T_{23}. \end{aligned}$$

To estimate T_{21}, we use again the fact that $\int_{\mathbb{R}^d} k_\varepsilon = 1$ to get

$$T_{21} \le \frac{M}{2}\int_{\mathbb{R}^d} \chi_R^c\,f^2\,m^2. \tag{5.7}$$

Then, to estimate T_{22}, we first perform a change of variable:

$$T_{22} = \frac{M}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} k(z)\,\chi_R^c(y+\varepsilon z)\,m^2(y+\varepsilon z)\,dz\,f^2(y)\,dy.$$

Using the mean value theorem, we deduce that there exist $\theta, \theta' \in (0,1)$ such that

$$\chi_R^c(y+\varepsilon z) = \chi_R^c(y) + \varepsilon\,z\cdot\nabla\chi_R^c(y+\theta\varepsilon z), \quad m^2(y+\varepsilon z) = m^2(y) + \varepsilon z\cdot\nabla m^2(y+\theta'\varepsilon z).$$

We then use the fact that $|\nabla\chi_R^c(y_0)| \le C_R$ where $C_R > 0$ is a constant which only depends on R. It implies that

$$T_{22} \le \frac{M}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} k(z)\,\left(\chi_R^c(y) + \varepsilon\,|z|\,C_R\right)\left(m^2(y) + \varepsilon z\cdot\nabla m^2(y+\theta'\,\varepsilon z)\right)\,dz\,f^2(y)\,dy.$$

Then, because of (5.1) and the fact that $|\nabla m^2(y+\theta'\varepsilon z)| \le C\,\langle y\rangle^{2q-1}\langle z\rangle^{2q-1}$, since $k \in L^1_{2q+1}$, we conclude that

$$T_{22} \le M\,C_R\,\kappa_\varepsilon \int_{\mathbb{R}^d} f^2\,m^2 + \frac{M}{2}\int_{\mathbb{R}^d} \chi_R^c\,f^2\,m^2, \quad \kappa_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0. \tag{5.8}$$

Putting together (5.7), (5.8) and the contribution of the term T_{23}, it yields

$$T_2 \le M\,C_R\,\kappa_\varepsilon \int_{\mathbb{R}^d} f^2\,m^2, \quad \kappa_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0. \tag{5.9}$$

As far as T_3 is concerned, we just perform an integration by parts:

$$\begin{aligned} T_3 &= \int_{\mathbb{R}^d} \operatorname{div}(xf)\,f\,m^2 \\ &= d\int_{\mathbb{R}^d} f^2\,m^2 + \int_{\mathbb{R}^d} x\cdot\nabla f\,f\,m^2 \\ &= d\int_{\mathbb{R}^d} f^2\,m^2 - \frac{1}{2}\int_{\mathbb{R}^d} f^2\,\operatorname{div}(x\,m^2) \\ &= \int_{\mathbb{R}^d} f^2(x)\,m^2(x)\left(\frac{d}{2} - \frac{q\,|x|^2\,\langle x\rangle^{2q-2}}{m^2(x)}\right)dx. \end{aligned} \tag{5.10}$$

The estimates (5.6), (5.9) and (5.10) together give

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f)\, f\, m^2 \le \int_{\mathbb{R}^d} f^2\, m^2 \left(C\, \langle x\rangle^{-2} + \frac{d}{2} - \frac{q\, |x|^2\, \langle x\rangle^{2q-2}}{m^2(x)} + M\, C_R\, \kappa_\varepsilon - M\, \chi_R \right)$$
$$= \int_{\mathbb{R}^d} f^2\, m^2 \left(\widetilde{\psi}_m + M\, C_R\, \kappa_\varepsilon - M\, \chi_R \right),$$

where we have denoted

$$\widetilde{\psi}_m(x) := C\, \langle x\rangle^{-2} + \frac{d}{2} - \frac{q\, |x|^2}{\langle x\rangle^2} \xrightarrow[|x|\to\infty]{} d/2 - q. \tag{5.11}$$

We recall that $M \ge 0$ and $R \ge 0$ are such that

$$\forall\, x \in \mathbb{R}^d, \quad \widetilde{\psi}_m(x) - M\, \chi_R(x) \le \frac{a}{2} + \frac{1}{2}\left(\frac{d}{2} - q\right).$$

We now pick $\varepsilon_0 \le \varepsilon_1$ small enough such that

$$\forall\, \varepsilon \in (0, \varepsilon_0], \quad M\, C_R\, \kappa_\varepsilon \le \frac{a}{2} - \frac{1}{2}\left(\frac{d}{2} - q\right).$$

This implies that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\psi_m - M\chi_R := \widetilde{\psi}_m + M\, C_R\, \kappa_\varepsilon - M\, \chi_R \le a.$$

As a conclusion, we obtain that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon - a)\, f\, f\, m^2 \le 0$$

for such a choice of M and R and we refer to [51, 64] for the proof in the case $\varepsilon = 0$. □

Lemma 5.2.5. *For any $a > -q$, there exist $\varepsilon_0 > 0$, $M \ge 0$ and $R \ge 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{B}_\varepsilon - a$ is dissipative in $L^1(m)$.*

Proof. We estimate the integral

$$\int_{\mathbb{R}^d} \mathcal{B}_\varepsilon f\, (\mathrm{sign} f)\, m$$
$$= \left(\frac{1}{\varepsilon^2} - M\right) \int_{\mathbb{R}^d} (k_\varepsilon * f - f)\, (\mathrm{sign} f)\, m + \int_{\mathbb{R}^d} M\, \chi_R^c\, (k_\varepsilon *_x f - f)\, (\mathrm{sign} f)\, m$$
$$+ \int_{\mathbb{R}^d} \mathrm{div}(x f)\, (\mathrm{sign} f)\, m - \int_{\mathbb{R}^d} M\, \chi_R\, |f|\, m$$
$$=: T_1 + T_2 + T_3 + T_4.$$

We omit the details of the proof which is very similar to the one of Lemma 5.2.4. We have

$$T_1 \le C \int_{\mathbb{R}^d} |f|(x)\, \langle x\rangle^{q-2}\, dx, \quad T_2 \le M\, C_R\, \kappa_\varepsilon \int_{\mathbb{R}^d} |f| m \quad \text{and} \quad T_3 = -\int_{\mathbb{R}^d} |f| m\, \frac{x \cdot \nabla m}{m}.$$

This implies that

$$\int_{\mathbb{R}^d} \mathcal{B}_\varepsilon f \,(\mathrm{sign} f)\, m \leq \int_{\mathbb{R}^d} |f|\, m \left(C \langle x \rangle^{-2} - \frac{x \cdot \nabla m}{m} + M\, C_R\, \kappa_\varepsilon - M\, \chi_R \right)$$

and we conclude as in the L^2 case. □

Lemma 5.2.6. *Let $s \in \mathbb{N}$ and $q > d/2 + s$. For any $a > d/2 - q + s$, there exist $\varepsilon_0 > 0$, $M \geq 0$ and $R \geq 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{B}_\varepsilon - a$ is hypodissipative in $H^s(m)$.*

Proof. The case $s = 0$ is nothing but Lemma 5.2.4. We now deal with the case $s = 1$. We consider f_t a solution to

$$\partial_t f_t = \mathcal{B}_\varepsilon f_t.$$

From the Lemma 5.2.4, we already know that

$$\frac{1}{2} \frac{d}{dt} \|f_t\|^2_{L^2(m)} \leq \int_{\mathbb{R}^d} f_t^2 \;\; m^2 \left(\widetilde{\psi}_m + M\, C_R \kappa_\varepsilon - M \chi_R \right). \tag{5.12}$$

We now want to compute the evolution of the derivative of f_t:

$$\partial_t \partial_x f_t = \mathcal{B}(\partial_x f_t) + M\, \partial_x(\chi_R^c)\,(k_\varepsilon * f_t - f_t) + \partial_x f_t,$$

which in turn implies that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\partial_x f_t\|^2_{L^2(m)} &= \int_{\mathbb{R}^d} (\partial_x f_t)\, \partial_t(\partial_x f_t)\, m^2 \\
&= \int_{\mathbb{R}^d} (\partial_x f_t)\, \mathcal{B}(\partial_x f_t)\, m^2 + \int_{\mathbb{R}^d} M\, \partial_x(\chi_R^c)\,(k_\varepsilon * f_t)\, \partial_x f_t\, m^2 \\
&\quad - \int_{\mathbb{R}^d} M\, \partial_x(\chi_R^c)\, f_t\, \partial_x f_t\, m^2 + \int_{\mathbb{R}^d} (\partial_x f_t)^2\, m^2 \\
&=: T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

Concerning T_1, using the proof of Lemma 5.2.4, we obtain

$$T_1 \leq \int_{\mathbb{R}^d} (\partial_x f_t)^2\, m^2 \left(\widetilde{\psi}_m + M\, C_R \kappa_\varepsilon - M\, \chi_R \right). \tag{5.13}$$

Then, to deal with T_2, we first notice that using Jensen inequality and (5.1), we have

$$\begin{aligned}
\|k_\varepsilon * f\|^2_{L^2(m)} &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k_\varepsilon(x-y)\, f(y)\, dy \right)^2 m^2(x)\, dx \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y)\, m^2(x)\, dx\, f^2(y)\, dy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} k(z)\, m^2(y + \varepsilon z)\, dz\, f^2(y)\, dy \\
&\leq C \int_{\mathbb{R}^d} k(z)\, m^2(z)\, dz \int_{\mathbb{R}^d} f^2\, m^2.
\end{aligned}$$

We thus obtain using that $k \in L^1_{2q}$:

$$\|k_\varepsilon * f\|_{L^2(m)} \le C\,\|f\|_{L^2(m)}.$$

The term T_2 is then treated using Cauchy-Schwarz inequality, Young inequality and the fact that $|\partial_x(\chi_R^c)|$ is bounded by a constant depending only on R:

$$\begin{aligned} T_2 &\le M\,C_R\,\|k_\varepsilon * f_t\|_{L^2(m)}\|\partial_x f_t\|_{L^2(m)} \\ &\le M\,C_R\,\|f_t\|_{L^2(m)}\|\partial_x f_t\|_{L^2(m)} \\ &\le M\,C_R\,\frac{1}{\beta}\|f_t\|^2_{L^2(m)} + M\,C_R\,\beta\|\partial_x f_t\|^2_{L^2(m)} \end{aligned} \tag{5.14}$$

for any $\beta > 0$ as small as we want.

The term T_3 is handled using an integration by parts and with the fact that $|\partial_x^2(\chi_R^c)|$ is bounded with a constant which only depends on R:

$$T_3 = \frac{M}{2}\int_{\mathbb{R}^d}\partial_x^2(\chi_R^c)\,f_t^2\,m^2 + \frac{M}{2}\int_{\mathbb{R}^d}\partial_x(\chi_R^c)\,f_t^2\,\partial_x(m^2) \le M\,C_R\,\|f_t\|^2_{L^2(m)}. \tag{5.15}$$

Finally, we have

$$T_4 = \|\partial_x f_t\|^2_{L^2(m)}. \tag{5.16}$$

Combining estimates (5.13), (5.14), (5.15) and (5.16), we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_x f_t\|^2_{L^2(m)} &\le C_{R,M,\beta}\int_{\mathbb{R}^d} f_t^2\,m^2 \\ &\quad + \int_{\mathbb{R}^d}(\partial_x f_t)^2\,m^2\,\left(\widetilde{\psi}_m + M\,C_R\,\kappa_\varepsilon + M\,C_R\,\beta + 1 - M\,\chi_R\right). \end{aligned} \tag{5.17}$$

To conclude the proof in the case $s = 1$, we introduce the norm

$$|||f|||^2_{H^1(m)} := \|f\|^2_{L^2(m)} + \eta\,\|\partial_x f\|^2_{L^2(m)}, \quad \eta > 0.$$

Combining (5.12) and (5.17), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|||f_t|||^2_{H^1(m)} &\le \int_{\mathbb{R}^d} f_t^2\,m^2\left(\widetilde{\psi}_m + M\,C_R\kappa_\varepsilon + \eta\,C_{R,M,\beta} - M\chi_R\right) \\ &\quad + \eta\int_{\mathbb{R}^d}(\partial_x f_t)^2\,m^2\,\left(\widetilde{\psi}_m + M\,C_R\,\kappa_\varepsilon + M\,C_R\,\beta + 1 - M\,\chi_R\right). \end{aligned}$$

Using the same strategy as in the proof of Lemma 5.2.4, if $a > d/2 - q + 1$, we can choose M, R large enough and β, ε_0, η small enough such that we have on $\mathbb{R}^d$

$$\widetilde{\psi}_m + M\,C_R\kappa_\varepsilon + \eta\,C_{R,M,\beta} - M\chi_R \le a \quad \text{and} \quad \widetilde{\psi}_m + M\,C_R\,\kappa_\varepsilon + M\,C_R\,\beta + 1 - M\,\chi_R \le a$$

for any $\varepsilon \in (0, \varepsilon_0]$, which implies that

$$\frac{1}{2}\frac{d}{dt}|||f_t|||^2_{H^1(m)} \le a\,|||f_t|||^2_{H^1(m)}.$$

To deal with the case $s = 2$, we have to write the evolution of $\partial_x^2 f_t$:

$$\partial_t \partial_x^2 f_t = \mathcal{B}(\partial_x^2 f_t) + 2M\,\partial_x(\chi_R^c)\,(k_\varepsilon *_x \partial_x f_t - \partial_x f_t) + 2\partial_x^2 f_t + M\,\partial_x^2\,(\chi_R^c)\,(k_\varepsilon * f - f)\,,$$

and then introduce the $H^2(m)$ norm defined by

$$|||f|||^2_{H^1(m)} = \|f\|^2_{L^2(m)} + \eta\,\|\partial_x f\|^2_{L^2(m)} + \eta^2\,\|\partial_x^2 f\|^2_{L^2(m)}, \quad \eta > 0.$$

The higher order derivatives are treated with the same method. □

5.2.5 Uniform regularization properties of $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$

We introduce the notation

$$I_\varepsilon(f) := \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (f(x) - f(y))^2\,k_\varepsilon(x-y)\,dx\,dy.$$

Lemma 5.2.7. *There exists a constant $K > 0$ such that for any $\varepsilon > 0$, the following estimate holds:*

$$\|k_\varepsilon *_x f\|^2_{\dot{H}^1} \le K\,I_\varepsilon(f). \tag{5.18}$$

Proof. First, performing a change of variable, one can notice that

$$\forall\,\xi \in \mathbb{R}^d, \quad \widehat{k_\varepsilon}(\xi) = \widehat{k}(\varepsilon\,\xi).$$

Using that $\int_{\mathbb{R}^d} k_\varepsilon = 1$, we have

$$\begin{aligned} I_\varepsilon(f) &= \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)^2\,k_\varepsilon(x-y)\,dx\,dy + \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(y)^2\,k_\varepsilon(x-y)\,dx\,dy \\ &\quad - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)f(y)\,k_\varepsilon(x-y)\,dx\,dy \\ &= \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}^d} f^2 - \int_{\mathbb{R}^d}(k_\varepsilon * f)\,f\right). \end{aligned}$$

As a consequence, using Plancherel formula, we get

$$I_\varepsilon(f) = \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}^d}\widehat{f}^2 - \int_{\mathbb{R}^d}\widehat{k_\varepsilon}\,\widehat{f}^2\right) = \int_{\mathbb{R}^d}\widehat{f}(\xi)^2\,\frac{1-\widehat{k}(\varepsilon\xi)}{\varepsilon^2}\,d\xi.$$

Then, we again use Plancherel formula to obtain

$$\begin{aligned} \|k_\varepsilon *_x f\|^2_{\dot{H}^1} &= \|\partial_x(k_\varepsilon *_x f)\|^2_{L^2} = \|\mathcal{F}(\partial_x(k_\varepsilon *_x f))\|^2_{L^2} \\ &= \int_{\mathbb{R}^d}|\xi|^2\,\widehat{k}(\varepsilon\xi)^2\,\widehat{f}^2. \end{aligned}$$

We conclude to (5.18) by using (5.3). □

We now introduce the following notation $\lambda := 1/(2K) > 0$. Before going into the proof of regularization lemmas, we recall a result from [63] which is going to be useful.

Lemma 5.2.8. *Consider two Banach spaces X, Y, two constants a_0, $b \in \mathbb{R}$, $a_0 < b$ and a function $u : \mathbb{R}^+ \to \mathscr{B}(X) + \mathscr{B}(Y)$ such that*

(1) $ue^{-at} \in L^1(0, \infty; \mathscr{B}(X) \cap \mathscr{B}(Y))$ for any $a > a_0$;

(2) $ue^{-bt} \in L^1(0, \infty; \mathscr{B}(X, Y))$.

Then, for any $a > a_0$, there exists $n \in \mathbb{N}$ such that $u^{(*n)} e^{-at} \in L^1(0, \infty; \mathscr{B}(X, Y))$, with explicit constant uniquely depending on the two assumed bounds (1) and (2).

Lemma 5.2.9. *Consider $s_1 < s_2 \in \mathbb{N}$ and $q > d/2 + s_2$. Let M, R and ε_0 so that the conclusion of Lemma 5.2.6 holds in both spaces $H^{s_1}(m)$ and $H^{s_2}(m)$. Then, for any $a \in (\max\{d/2 - q + s_2, -\lambda\}, 0)$, there exists $\ell \in \mathbb{N}$ such that for any $\varepsilon \in (0, \varepsilon_0]$, we have the following estimate*

$$\int_0^\infty \| \, (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t) \|_{H^{s_1}(m) \to H^{s_2}(m)} \, e^{-at} \, dt \leq C_a$$

for some constant $C_a > 0$.

Proof. We first give the proof for the case $(s_1, s_2) = (0, 1)$. We consider $a \in (\max\{d/2 - q + 1, -\lambda\}, 0)$, $b \in (\max\{d/2 - q + 1, -\lambda\}, a)$ and $f_t := S_{\mathcal{B}_\varepsilon}(t) f$, i.e. that satisfies

$$\partial_t f_t = \mathcal{B}_\varepsilon f_t, \quad f_0 = f.$$

From the proof of Lemma 5.2.6, for any $\varepsilon \in (0, \varepsilon_0]$, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|f_t\|^2_{L^2(m)} \\
&\leq -\frac{1}{2} \left(\frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))^2 \, k_\varepsilon(x-y) \, m^2(x) \, dy \, dx + a \, \|f_t\|^2_{L^2(m)} \\
&\leq -\frac{1}{4\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))^2 \, k_\varepsilon(x-y) \, dy \, dx + a \, \|f_t\|^2_{L^2(m)} \\
&\leq -\frac{1}{2} \, I_\varepsilon(f_t) + a \, \|f_t\|^2_{L^2(m)}
\end{aligned}$$

where we have used that $M \leq 1/(2\varepsilon^2)$ for any $\varepsilon \in (0, \varepsilon_0]$. Using Lemma 5.2.7, we obtain

$$\begin{aligned}
\frac{d}{dt} \|f_t\|^2_{L^2(m)} &\leq -2\lambda \|k_\varepsilon *_x f_t\|^2_{\dot{H}^1} + 2a \, \|f_t\|^2_{L^2(m)} \\
&\leq 2a \, \|k_\varepsilon *_x f_t\|^2_{\dot{H}^1} + 2a \, \|f_t\|^2_{L^2(m)}.
\end{aligned}$$

Multiplying this inequality by e^{-2at}, it implies that

$$\frac{d}{dt} \left(\|f_t\|^2_{L^2(m)} \, e^{-2at} \right) \leq 2a \, \|k_\varepsilon *_x f_t\|^2_{\dot{H}^1} \, e^{-2at}$$

and thus, integrating in time

$$\|f_t\|^2_{L^2(m)}\, e^{-2at} - 2a \int_0^t \|k_\varepsilon *_x f_s\|^2_{\dot H^1} e^{-2as}\, ds \le \|f\|^2_{L^2(m)}.$$

In particular, we obtain

$$\int_0^t \|k_\varepsilon *_x f_s\|^2_{\dot H^1} e^{-2as}\, ds \le -\frac{1}{2a}\, \|f\|^2_{L^2(m)}, \quad \forall\, t \ge 0. \tag{5.19}$$

We now want to estimate

$$\begin{aligned}
&\int_0^t \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^1(m)}\, e^{-2as}\, ds = \int_0^t \|\mathcal{A}_\varepsilon f_s\|^2_{H^1(m)}\, e^{-2as}\, ds \\
&= \int_0^t \|\mathcal{A}_\varepsilon f_s\|^2_{L^2(m)}\, e^{-2as}\, ds + \int_0^t \|\partial_x \left(\mathcal{A}_\varepsilon f_s\right)\|^2_{L^2(m)}\, e^{-2as}\, ds \\
&\le \int_0^t \|\mathcal{A}_\varepsilon f_s\|^2_{L^2(m)}\, e^{-2as}\, ds + \int_0^t \|M \partial_x(\chi_R)\, k_\varepsilon *_x f_s\|^2_{L^2(m)}\, e^{-2as}\, ds \\
&\quad + \int_0^t \|M \chi_R\, \partial_x(k_\varepsilon *_x f_s)\|^2_{L^2(m)}\, e^{-2as}\, ds \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}$$

Using dissipativity properties of $\mathcal{B}_\varepsilon$ and boundedness of $\mathcal{A}_\varepsilon$, we get

$$I_1 \le C \int_0^t e^{2bs} e^{-2as}\, ds\, \|f\|^2_{L^2(m)} \le C\, \|f\|^2_{L^2(m)}.$$

We deal with I_2 using the fact that $M\partial_x(\chi_R)$ is compactly supported, Young inequality and dissipativity properties of $\mathcal{B}_\varepsilon$:

$$I_2 \le C \int_0^t \|k_\varepsilon *_x f_s\|^2_{L^2}\, ds \le C \int_0^t \|f_s\|^2_{L^2} ds \le C \int_0^t e^{2bs}\, ds\, \|f\|^2_{L^2(m)} \le C\, \|f\|^2_{L^2(m)}.$$

Finally, for I_3, we use (5.19) to obtain

$$I_3 \le \int_0^t \|\; k_\varepsilon *_x f_s\|^2_{\dot H^1}\, e^{-2as}\, ds \le C\, \|f\|^2_{L^2(m)}.$$

Passing to the limit $t \to \infty$, we obtain

$$\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^1(m)}\, e^{-2as}\, ds \le C\, \|f\|^2_{L^2(m)}.$$

Consequently, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left(\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|_{H^1(m)}\, e^{-as/2}\, ds\right)^2 &= \left(\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|_{H^1(m)}\, e^{-as}\, e^{as/2}\, ds\right)^2 \\
&\le \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^1(m)}\, e^{-2as}\, ds \int_0^\infty e^{as}\, ds \\
&\le C\, \|f\|^2_{L^2(m)}.
\end{aligned} \tag{5.20}$$

To conclude the proof in the case $(s_1, s_2) = (0, 1)$, we use Lemma 5.2.8 with $X = L^2(m)$, $Y = H^1(m)$ and $u(t) = \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$. Lemmas 5.2.3 and 5.2.4-5.2.6 allow us to check that assumptions (1) is satisfied and assumption (2) comes from (5.20).

Using the same strategy, we can easily obtain that

$$\int_0^\infty \| \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^2(m)}\, e^{-2as}\, ds \le C\, \|f\|^2_{H^1(m)}.$$

We can thus deduce that

$$\begin{aligned}
&\int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*2)}(t) f\|^2_{H^2(m)}\, e^{-2at}\, dt \\
&\le \int_0^\infty \int_0^t \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t-s) \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^2(m)}\, e^{-2a(t-s)}\, e^{-2as}\, ds\, dt \\
&\le \int_0^\infty \int_s^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t-s) \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^2(m)}\, e^{-2a(t-s)}\, dt\, e^{-2as}\, ds \\
&\le \int_0^\infty \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t) \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^2(m)}\, e^{-2at}\, dt\, e^{-2as}\, ds \\
&\le C \int_0^t \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|^2_{H^1(m)}\, e^{-2as}\, ds \\
&\le C\, \|f\|^2_{L^2(m)}.
\end{aligned}$$

Reiterating the process, we can conclude the proof of the lemma. □

Lemma 5.2.10. *Consider $q > d/2$ and M, R, ε_0 so that the conclusions of Lemmas 5.2.4 and 5.2.5 hold. Then, for any $a \in (-q, 0)$ there exists $n \in \mathbb{N}$ such that the following estimate holds for any $\varepsilon \in (0, \varepsilon_0]$:*

$$\forall\, t \ge 0, \quad \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{\mathscr{B}(L^1(m), L^2(m))}\, e^{-at}\, dt \le C_a,$$

for some constant $C_a > 0$.

Proof. We first introduce the formal dual operators of $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$:

$$\mathcal{A}^*_\varepsilon \phi := k_\varepsilon * (M\, \chi_R\, \phi), \quad \mathcal{B}^*_\varepsilon \phi := \frac{1}{\varepsilon^2}(k_\varepsilon * \phi - \phi) - x \cdot \nabla \phi - k_\varepsilon * (M\, \chi_R \phi).$$

We use the same computation as the one used to deal with T_1 is the proof of Lemma 5.2.4 and Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_{\mathbb{R}^d} (\mathcal{B}^*_\varepsilon \phi)\, \phi \le &-\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y)\, (\phi(y) - \phi(x))^2\, dy\, dx \\
&+ \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi^2(y) - \phi^2(x))\, k_\varepsilon(x-y)\, dy\, dx \\
&+ \frac{d}{2} \int_{\mathbb{R}^d} \phi^2 + \|k_\varepsilon * (M\, \chi_R\, \phi)\|_{L^2}\, \|\phi\|_{L^2}.
\end{aligned}$$

We then notice that the second term equals 0 and we use Young inequality and the fact that $\|k_\varepsilon\|_{L^1} = 1$ to get

$$\begin{aligned}\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^* \phi)\, \phi &\le -\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d\times\mathbb{R}^d} k_\varepsilon(x-y)\, (\phi(y)-\phi(x))^2\, dy\, dx \\ &\quad + \frac{d}{2} \int_{\mathbb{R}^d} \phi^2 + \frac{1}{2} \|M\, \chi_R\, \phi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^2}^2 \\ &\le -\, I_\varepsilon(\phi) + C \int_{\mathbb{R}^d} \phi^2.\end{aligned}$$

We also have the following inequality:

$$\begin{aligned}I_\varepsilon(\chi_R\, \phi) &\le \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d\times\mathbb{R}^d} k_\varepsilon(x-y)\, \phi^2(x)\, (\chi_R(y)-\chi_R(x))^2\, dy\, dx \\ &\quad + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d\times\mathbb{R}^d} k_\varepsilon(x-y)\, \chi_R^2(y)\, (\phi(y)-\phi(x))^2\, dy\, dx \\ &\le C\, \|\nabla \chi_R\|_\infty \int_{\mathbb{R}^d} \phi^2 + 2 I_\varepsilon(\phi).\end{aligned}$$

If we denote $\phi_t := S_{\mathcal{B}_\varepsilon^*}(t)\phi$, we thus have

$$\frac{1}{2}\frac{d}{dt} \|\phi_t\|_{L^2}^2 \le -\lambda\, \|k_\varepsilon * (\chi_R\, \phi_t)\|_{\dot H^1}^2 + b\, \|\phi_t\|_{L^2}^2, \quad b > 0.$$

Multiplying this inequality by e^{-bt}, we obtain

$$\frac{d}{dt} \left(\|\phi_t\|_{L^2}^2\, e^{-bt} \right) \le -2\lambda\, \|k_\varepsilon * (\chi_R\, \phi_t)\|_{\dot H^1}^2\, e^{-bt}, \qquad \forall\, t \ge 0,$$

and integrating in time, we get

$$\|\phi_t\|_{L^2}^2\, e^{-bt} + 2\lambda \int_0^t \|k_\varepsilon * (\chi_R\, \phi_s)\|_{\dot H^1}^2\, e^{-bs}\, ds \le \|\phi\|_{L^2(m)}^2, \quad \forall\, t \ge 0. \tag{5.21}$$

We now estimate

$$\begin{aligned}\int_0^t \|\mathcal{A}_\varepsilon^*\, S_{\mathcal{B}_\varepsilon^*}(s)\, \phi\|_{H^1}^2\, e^{-2bs}\, ds &= \int_0^t \|\mathcal{A}_\varepsilon^*\, \phi_s\|_{H^1}^2\, e^{-2bs}\, ds \\ = \int_0^t \|k_\varepsilon * (M\, \chi_R\, \phi_s)\|_{L^2}^2\, e^{-2bs}\, ds &+ \int_0^t \|k_\varepsilon * (M\, \chi_R\, \phi_s)\|_{\dot H^1}^2\, e^{-2bs}\, ds.\end{aligned}$$

Using Young inequality and (5.21), we conclude that

$$\int_0^\infty \|\, \mathcal{A}_\varepsilon^*\, S_{\mathcal{B}_\varepsilon^*}(t)\, \phi\|_{H^1}^2\, e^{-2bs}\, ds \le C\, \|\phi\|_{L^2}^2.$$

As in the proof of Lemma 5.2.9, we can obtain that for any $s \in \mathbb{N}$, there exists a constant $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\int_0^\infty \|(\mathcal{A}_\varepsilon^*\, S_{\mathcal{B}_\varepsilon^*})^{(*s)}(t)\|_{L^2 \to H^s}^2\, e^{-2bt}\, dt \le C.$$

From this, we deduce that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\int_0^\infty \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*s)}(t)\|^2_{H^{-s}\to L^2} \, e^{-2bt} \, dt \le C.$$

Taking $\ell > d/2$ and using the continuous Sobolev embedding $L^1(\mathbb{R}^d) \subset H^{-\ell}(\mathbb{R}^d)$, we obtain

$$\int_0^\infty \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)}(t)\|^2_{L^1\to L^2} \, e^{-2bt} \, dt \le C.$$

The integer ℓ is thus fixed such that $\ell > d/2$. Then noticing that

$$(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*(\ell+1))} = \mathcal{A}_\varepsilon \, (S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)} * S_{\mathcal{B}_\varepsilon}$$

and using the fact that $\mathcal{A}_\varepsilon$ is compactly supported combined with Lemma 5.2.5, we get

$$\begin{aligned}
&\int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*(\ell+1))} f\|^2_{L^2(m)} \, e^{-2bt} \, dt \\
&\le \int_0^\infty \int_0^t \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)}(s) \, S_{\mathcal{B}_\varepsilon}(t-s) f\|^2_{L^2} \, e^{-2bt} \, dt \\
&\le \int_0^\infty \int_s^\infty \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)}(s)\|^2_{L^1\to L^2} \, e^{-2bs} \, \|S_{\mathcal{B}_\varepsilon}(t-s) f\|^2_{L^1(m)} e^{-2b(t-s)} \, dt \, ds \\
&\le \int_0^\infty \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)}(s)\|^2_{L^1\to L^2} \, e^{-2bs} \, ds \int_0^\infty e^{2(a-b)t} \, dt \, \|f\|^2_{L^1(m)} \\
&\le C \, \|f\|^2_{L^1(m)}.
\end{aligned}$$

Consequently, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left(\int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*(\ell+1))} f\|_{L^2(m)} \, e^{-2bt} \, dt\right)^2 \\
&\le \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*(\ell+1))} f\|^2_{L^2(m)} \, e^{-2bt} \, dt \int_0^\infty e^{-2bt} \, dt \\
&\le C \, \|f\|^2_{L^1(m)}.
\end{aligned} \tag{5.22}$$

To conclude the proof, we use Lemma 5.2.8 with $X = L^1(m)$, $Y = L^2(m)$ and u is the function $u(t) := (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*(\ell+1))}(t)$. We are able to check that assumption (1) is satisfied thanks to Lemmas 5.2.3 and 5.2.4-5.2.5. Assumption (2) is nothing but inequality (5.22). □

5.2.6 Spectral analysis

Lemma 5.2.11. *For any $\varepsilon > 0$, Λ_ε satisfies Kato's inequalities:*

$$\forall\, f \in D(\Lambda_\varepsilon), \quad \Lambda_\varepsilon\,(\theta(f)) \ge \theta'(f)\,(\Lambda_\varepsilon f), \quad \theta(s) = |s| \quad \textit{or} \quad \theta(s) = s_+.$$

It follows that for any $\varepsilon > 0$, the semigroup associated to Λ_ε is positive in the following sense that if $f \in L^1(m)$ and $f \ge 0$, then for any $t \ge 0$, $S_{\Lambda_\varepsilon}(t) f \ge 0$.

Proof. First, we have

$$\begin{aligned}\mathrm{sign} f(x)\,\Lambda_\varepsilon f(x) &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} k_\varepsilon(x-y)\,f(y)\,dy\,\mathrm{sign} f(x) + d\,f(x)\,\mathrm{sign} f(x)\\ &\quad + x\cdot\nabla f(x)\,\mathrm{sign} f(x)\\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} k_\varepsilon(x-y)\,|f|(y)\,dy + d\,|f|(x) + x\cdot\nabla|f|(x) = \Lambda_\varepsilon|f|(x),\end{aligned}$$

which ends the proof of the Kato inequality in the case $\theta(s) = |s|$. Using that $s_+ = (s+|s|)/2$, we obtain the result in the case $\theta(s) = s_+$.

We consider $f \le 0$ and denote $f(t) := S_{\Lambda_\varepsilon}(t)f$. We define $\beta(s) = s_+ = (|s|+s)/2$. Using Kato's inequality, we have $\partial_t \beta(f_t) \le \Lambda_\varepsilon \beta(f_t)$, and then

$$0 \le \int_{\mathbb{R}^d} \beta(f_t) \le \int_{\mathbb{R}^d} \beta(f) = 0, \quad \forall\, t \ge 0,$$

from which we deduce $f_t \le 0$ for any $t \ge 0$. □

The operator $-\Lambda_\varepsilon$ satisfies the following form of the strong maximum principle.

Lemma 5.2.12. *Any nonnegative eigenfunction associated to the eigenvalue* 0 *is positive. In other words, we have*

$$f \in D(\Lambda_\varepsilon), \quad \Lambda_\varepsilon f = 0, \quad f \ge 0, \quad f \not\equiv 0 \quad \textit{implies} \quad f > 0.$$

Proof. We define

$$\mathcal{C}f = \frac{1}{\varepsilon^2}\,k_\varepsilon * f, \quad \mathcal{D}f = x\cdot\nabla_x f + \lambda\, f, \quad \lambda := d - \frac{1}{\varepsilon^2}$$

and the semigroup

$$S_{\mathcal{D}}(t)g := g(e^t\,x)\,e^{\lambda t}$$

with generator $\mathcal{D}$. Thanks to the Duhamel formula

$$S_{\Lambda_\varepsilon}(t) = S_{\mathcal{D}}(t) + \int_0^t S_{\mathcal{D}}(s)\mathcal{C}S_\Lambda(t-s)\,ds,$$

the eigenfunction f satisfies

$$\begin{aligned} f &= S_{\Lambda_\varepsilon}(t)f = S_{\mathcal{D}}(t)f + \int_0^t S_{\mathcal{D}}(s)\mathcal{C}S_{\Lambda_\varepsilon}(t-s)f\,ds\\ &\ge \int_0^t S_{\mathcal{D}}(s)\mathcal{C}f\,ds \quad \forall\, t > 0.\end{aligned}$$

By assumption, there exists $x_0 \in \mathbb{R}^d$ such that $f \not\equiv 0$ on $B(x_0, r/2)$. As a consequence, denoting $\rho := \|\,f\|_{L^1(B(x_0,r/2))} > 0$, we have

$$\mathcal{C}f \ge \frac{\kappa\,\rho}{\varepsilon^2}\,\mathbf{1}_{B(x_0,r/2)},$$

and then

$$f \geq \frac{\kappa\,\rho}{\varepsilon^2} \sup_{t>0} \int_0^t e^{\lambda s}\, \mathbf{1}_{B(e^{-s}\ x_0, e^{-t}\ r/2)}\, ds \geq \kappa_1 \mathbf{1}_{B(x_0, r/4)}, \quad \kappa_1 > 0.$$

Using that lower bound, we obtain

$$\mathcal{C} f \geq \theta_d \frac{\kappa\,\kappa_{i-1}}{\varepsilon^2}\, \mathbf{1}_{B(x_0, u_i r)}, \quad \text{and then } f \geq \kappa_i \mathbf{1}_{B(x_0, v_i r)},$$

with $i = 2$, $u_2 = 1$, $\kappa_2 > 0$, $u_2 := 3/4$. Repeating once more the argument, we get the same lower estimate with $i = 3$, $u_3 = 7/4$, $\kappa_3 > 0$ and $v_3 = 3/2$. By an induction argument, we finally get $f > 0$ on $\mathbb{R}^d$. □

We are now able to prove Theorem 5.2.1.

Proof of part (1) in Theorem 5.2.1. Using Lemmas 5.2.3-5.2.5-5.2.6, 5.2.11, 5.2.12 and the fact that $\Lambda_\varepsilon^* 1 = 0$, we can apply Krein-Rutman theorem which implies that for any $\varepsilon > 0$, there exists a unique $G_\varepsilon > 0$ such that $\|G_\varepsilon\|_{L^1} = 1$, $\Lambda_\varepsilon G_\varepsilon = 0$ and $\Pi_\varepsilon f = \langle f \rangle G_\varepsilon$ where $\langle f \rangle = \int_{\mathbb{R}^d} f$. It also implies that if $X = L^1(m)$ or $X = H^s(m)$ for any $s \in \mathbb{N}$, for any $\varepsilon > 0$, there exists $a_\varepsilon < 0$ such that in X, there holds

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_\varepsilon} = \{0\}$$

and

$$\forall\, t \geq 0, \quad \|S_{\Lambda_\varepsilon}(t) f - \langle f \rangle\, G_\varepsilon\|_X \leq e^{at} \|f - \langle f \rangle\, G_\varepsilon\|_X, \quad \forall\, a > a_\varepsilon. \tag{5.23}$$

Proof of part (2) in Theorem 5.2.1. We now have to establish that estimate (5.23) can be obtained uniformly in $\varepsilon \in [0, \varepsilon_0]$. In order to do so, we use a perturbation argument in the same line as in [66, 88] to prove that our operator Λ_ε has a spectral gap in $H^3(m)$ which does not depend on ε.

First, we introduce the following spaces:

$$X_1 := H^6_1(m) \subset X_0 := H^3(m) \subset X_{-1} := L^2(m)$$

where $m = \langle x \rangle^q$ with $q > d/2 + 5$ so that the conclusion of Lemma 5.2.6 is satisfied in the three spaces X_i, $i = -1, 0, 1$.

One can notice that we also have the following embedding

$$X_1 \subset H^5_1(m) \subset D(\Lambda_\varepsilon) = D(\mathcal{B}_\varepsilon) \subset D(\mathcal{A}_\varepsilon) \subset X_0.$$

We now summarize the necessary results to apply a perturbative argument (obtained thanks to Lemmas 5.2.2, 5.2.3, 5.2.4, 5.2.6 and 5.2.9 and from [51, 64]).

There exist $a_0 < 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$:

(i) For any $i = -1, 0, 1$, $\mathcal{A}_\varepsilon \in \mathscr{B}(X_i)$ uniformly in ε.

(ii) For any $a > a_0$ and $\ell \geq 0$, there exists $C_{\ell,a} > 0$ such that

$$\forall\, i = -1, 0, 1, \quad \forall\, t \geq 0, \quad \|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{X_i \to X_i} \leq C_{\ell,a}\, e^{at}.$$

(iii) For any $a > a_0$, there exist $n \geq 1$ and $C_{n,a} > 0$ such that

$$\forall\, i = -1, 0, \quad \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{X_i \to X_{i+1}}\, e^{-at}\, dt \leq C_{n,a}.$$

(iv) There exists a function $\eta(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$ such that

$$\forall\, i = -1, 0, \quad \|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \to X_i} \leq \eta(\varepsilon) \quad \text{and} \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \to X_{i-1}} \leq \eta(\varepsilon).$$

(v) $\Sigma(\Lambda_0) \cap \Delta_{a_0} = \{0\}$ in spaces X_i, $i = -1, 0, 1$, where 0 is a one dimensional eigenvalue.

Using a perturbative argument, from the facts (i)–(v), we can deduce the following proposition:

Proposition 5.2.13. *There exist $a_0 < 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, the following properties hold in $X_0 = H^3(m)$:*

1. $\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_0} = \{0\}$;
2. *for any $f \in X_0$ and any $a > a_0$,*

$$\|S_{\Lambda_\varepsilon}(t) f - G_\varepsilon \langle f \rangle\|_{X_0} \leq C_a\, e^{at}\, \|f - G_\varepsilon \langle f \rangle\|_{X_0}, \quad \forall\, t \geq 0$$

for some explicit constant $C_a > 0$.

To end the proof of Theorem 5.2.1, we enlarge the space where the previous estimates hold. To do that, we use an extension argument (see [51, 66]) and Lemmas 5.2.3, 5.2.5-5.2.6 and 5.2.9-5.2.10. Our "small space" is $H^3(m)$ and our "large" space is $L^1(m)$.

5.3 From fractional to classical Fokker-Planck equation

We introduce the polynomial weight $m(x) := \langle x \rangle^q$ with $0 < q < 2$. In this part, we denote $\alpha := 2 - \varepsilon \in (0, 2]$ and we deal with the equations

$$\begin{cases} \partial_t f = -(-\Delta)^{\alpha/2} f + \operatorname{div}(xf) = \Lambda_{2-\alpha} f =: \mathcal{L}_\alpha f, \quad \alpha \in (0,2) \\ \partial_t f = \Delta f + \operatorname{div}(xf) = \Lambda_0 f =: \mathcal{L}_2 f. \end{cases} \tag{5.24}$$

We here recall (see (5.2)) that for $\alpha \in (0, 2)$, the fractional Laplacian of Schwartz function is defined using an integral formulation as follows:

$$\forall\, f \in \mathcal{S}(\mathbb{R}^d), \quad (-\Delta)^{\alpha/2} f(x) := c_\alpha \int_{\mathbb{R}^d} \frac{f(x) - f(y) + \chi(x-y)(x-y) \cdot \nabla f(x)}{|x-y|^{d+\alpha}}\, dy,$$

where $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$. Moreover, c_α is a constant depending on α which satisfies

$$\frac{c_\alpha}{2} \int_{|z| \leq i} \frac{z_i^2}{|z|^{d+\alpha}} = 1, \quad \forall\, i = 1, \ldots, d,$$

which implies that $c_\alpha \approx (2-\alpha)$.

We recall that the equation $\partial_t f = \mathcal{L}_\alpha f$ admits a unique equilibrium of mass 1 that we denote G_α (see [45] for example). Moreover, if $\alpha < 2$, one can prove that $G_\alpha(x) \approx \langle x\rangle^{-d-\alpha}$ (see [90]) and for $\alpha = 2$, we have $G_2(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$. The main result of this section reads:

Theorem 5.3.14. *Assume $\alpha_0 \in (0,2)$ and $q < \alpha_0$. There exists an explicit constant $a_0 < 0$ such that for any $\alpha \in [\alpha_0, 2]$, the semigroup $S_{\mathcal{L}_\alpha}(t)$ associated to the fractional Fokker-Planck equation (5.24) satisfies: for any $f \in L^1_q$, any $a > a_0$ and any $\alpha \in [\alpha_0, 2]$,*

$$\|S_{\mathcal{L}_\alpha}(t) f - G_\alpha \langle f\rangle\|_{L^1_q} \le C_a e^{at} \|f - G_\alpha\langle f\rangle\|_{L^1_q}$$

for some explicit constant $C_a > 0$. In particular, the spectrum $\Sigma(\mathcal{L}_\alpha)$ of $\mathcal{L}_\alpha$ satisfies the separation property $\Sigma(\mathcal{L}_\alpha) \cap \Delta_{a_0} = \{a_0\}$ in $L^1(\langle x\rangle^q)$ for any $\alpha \in [\alpha_0, 2]$.

5.3.1 Exponential decay in $L^2(G_\alpha^{-1/2})$

We recall a result from [45] which establishes an exponential decay to equilibrium for the semigroup $S_{\mathcal{L}_\alpha}(t)$.

Theorem 5.3.15. *There exists a constant $a_0 < 0$ such that for any $\alpha \in (0,2)$, we have the following estimate:*

$$\|S_{\mathcal{L}_\alpha}(t) f - \langle f\rangle G_\alpha\|_{L^2(G_\alpha^{-1/2})} \le e^{a_0 t} \|f - \langle f\rangle G_\alpha\|_{L^2(G_\alpha^{-1/2})}, \quad \forall\, t \ge 0.$$

Proof. The proof is immediate going back to the proof of the exponential decay in the space $L^2(G_\alpha^{-1/2})$ from [45]. Indeed, we can notice that the rate of decrease can be made uniform in α. □

5.3.2 Splitting of $\mathcal{L}_\alpha$

We define $\mathcal{A}_\alpha := M\,\chi_R$ and $\mathcal{B}_\alpha := \mathcal{L}_\alpha - \mathcal{A}_\alpha$ for some $M, R > 0$ to be chosen later.

5.3.3 Uniform boundedness of $\mathcal{A}_\alpha$

Lemma 5.3.16. *Consider $s \in \mathbb{N}$ and $p \ge 1$. The operator is uniformly bounded in α from $W^{s,p}(\omega)$ to $W^{s,p}$ with $\omega = m$ or $\omega = G_\alpha^{-1/2}$.*

Proof. The proof is immediate using that $M\,\chi_R$ and all its derivatives are compactly supported. □

5.3.4 Uniform dissipativity properties of $\mathcal{B}_\alpha$

Lemma 5.3.17. *For any $a > -q$, there exist $M > 0$ and $R > 0$ such that for any $\alpha \in [\alpha_0, 2]$, $\mathcal{B}_\alpha - a$ is dissipative in $L^1(m)$.*

Proof. We just have to adapt the proof Lemma 5.1 from [90] taking into account the constant c_α. Indeed, we have

$$\int_{\mathbb{R}^d} (\mathcal{L}_\alpha f)\, \mathrm{sign} f\, m \le \int_{\mathbb{R}^d} |f|\, m \left(\frac{I_\alpha(m)}{m} - \frac{x \cdot \nabla m}{m} \right).$$

We can then show that thanks to the rescaling constant c_α, $I_\alpha(m)/m$ goes to 0 at infinity uniformly in $\alpha \in [\alpha_0, 2)$. As a consequence, if $a > -q$, since $(x \cdot \nabla m)/m$ goes to $-q$ ate infinity, one may choose M and R such that for any $\alpha \in [\alpha_0, 2)$,

$$\frac{I_\alpha(m)}{m} - \frac{x \cdot \nabla m}{m} - M\, \chi_R \le a, \quad \text{on } \mathbb{R}^d,$$

which gives the result.

□

Lemma 5.3.18. *For any $a > a_0$ where a_0 is defined in Theorem 5.3.15, $\mathcal{B}_\alpha - a$ is dissipative in $L^2(G_\alpha^{-1/2})$.*

Proof. The proof also comes from the one of [90, Lemma 5.1]. □

5.3.5 Uniform regularization properties of $\mathcal{A}_\alpha S_{\mathcal{B}_\alpha}(t)$

Lemma 5.3.19. *There exist some constants $b \in \mathbb{R}$ and $C > 0$ such that for any $\alpha \in [\alpha_0, 2]$, the following estimate holds:*

$$\forall\, t \ge 0, \quad \|S_{\mathcal{B}_\alpha}(t)\|_{\mathscr{B}(L^1, L^2)} \le C\, \frac{e^{bt}}{t^{d/2\alpha_0}}.$$

As a consequence, we can prove that for any $a > \max(-q, a_0)$ and any $\alpha \in [\alpha_0, 2]$,

$$\forall\, t \ge 0, \quad \|(\mathcal{A}_\alpha\, S_{\mathcal{B}_\alpha})^{(*n)}(t)\|_{\mathscr{B}(L^1(m), L^2(G_\alpha^{-1/2}))} \le e^{at}. \tag{5.25}$$

Proof. We do not write the proof for the case $\alpha = 2$ and refer to [51, 64].
Step 1. The key argument to prove this regularization property of $S_{\mathcal{B}}(t)$ is the Nash inequality. For $\alpha \in [\alpha_0, 2)$, from the proof of [90, Lemma 5.3], we obtain that there exist $b \ge 0$ and $C > 0$ such that for any $\alpha \in [\alpha_0, 2)$,

$$\forall\, t \ge 0, \quad \|S_{\mathcal{B}_\alpha}(t) f\|_{L^2} \le C\, \frac{e^{bt}}{t^{d/(2\alpha_0)}}\, \|f\|_{L^1}.$$

Step 2. Then, using that $\mathcal{A}_\alpha$ is compactly supported, we can write

$$\|\mathcal{A}_\alpha S_{\mathcal{B}_\alpha}(t) f\|_{L^2(m)} \le C\, \|S_{\mathcal{B}_\alpha}(t) f\|_{L^2} \le C\, \frac{e^{bt}}{t^{d/(2\alpha_0)}}\, \|f\|_{L^1}.$$

Using the same method as in [51], we can first deduce that there exists $\ell_0 \in \mathbb{N}$, $\gamma \in [0,1)$ and $K \in \mathbb{R}$ such that for any $\alpha \in [\alpha_0, 2]$,

$$\|(\mathcal{A}_\alpha S_{\mathcal{B}_\alpha})^{(*\ell_0)}(t) f\|_{L^2(G_\alpha^{-1/2})} \le C\, \frac{e^{bt}}{t^\gamma}\, \|f\|_{L^1(m)}.$$

We can then conclude that (5.25) holds using [51, Lemma 2.17] combined with Lemmas 5.3.17 and 5.3.16. □

5.3.6 Spectral analysis

Before going into the proof of Theorem 5.3.14, let us notice that we can make explicit the projection Π_α onto the null space $\mathcal{N}(\mathcal{L}_\alpha)$ through the following formula: $\Pi_\alpha f = \langle f \rangle \, G_\alpha$. Moreover, since the mass is preserved by the equation $\partial_t f = \mathcal{L}_\alpha f$, we can deduce that $\Pi_\alpha(S_{\mathcal{L}_\alpha}(t) f) = \Pi_\alpha f$ for any $t \geq 0$.

Proof of Theorem 5.3.14. We can apply [51, Theorem 2.13] for each $\alpha \in [\alpha_0, 2]$ because combining Theorem 5.3.15 with Lemmas 5.3.16, 5.3.17, 5.3.18 and 5.3.19, we can check the assumptions of the theorem are satisfied. This gives us the conclusion of the theorem. □

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